Engineering optimization.
Chapter 7: Constrained direct search

Huy-Dung Han, Fabio E. Lapicccirella
Department of Electrical and Computer Engineering
University of California, Davis
Davis, CA 95616, U.S.A.
Advisors: Professor Xin Liu and Professor Zhi Ding
Professor Biswanath Mukherjee’s group study

July 16, 2010
Outline

1. Introduction.
2. Problem preparation
3. Feasible starting points generation
4. Unconstrained search methods adaptation
5. Complex method
6. Random search
7. Conclusions
Introduction

Scope: *constrained-optima localization*

Assumptions:
- Only function-values available.
- No gradient information.
- No analysis possible.

Difference from previous chapters:
- Explicit constraints consideration.
Two families of methods:
- Adaptation of unconstrained direct-search methods.
- Random selection of trail points.

Characteristics:
- Heuristic methods.
- Easy to implement.
- Slow convergence to optima.
- No guarantees on global optima.
Outline

1. Introduction.
2. Problem preparation
3. Feasible starting points generation
4. Unconstrained search methods adaptation
5. Complex method
6. Random search
7. Conclusions
Intuition:

- Given an initial feasible point $x^0$.
- Modify function-value based unconstrained localization methods to satisfy problem constraints.

Equality constraints issue:

- Assume points $x^0$ and $v$ satisfy constraints.
- Point: $x = x^0 + \alpha(v - x^0), \text{with } 0 \leq \alpha \leq 1$, need not satisfy constraints.

All direct-search methods adaptation require problems posed only in terms of inequality constraints.
Equality constraints treatment

- Eliminate equality constraints by *substitution*.

Example 7.1

Minimize \( f(\mathbf{x}) = x_1^2 + 4x_2^2 + x_3^2 \)

Subject to: \( h_1(\mathbf{x}) = x_1 + x_2 - 2 = 0 \)

\(-1 \leq x_1 \leq 1\)

\(0 \leq (x_2, x_3) \leq 2.\) (1)

Substitution of \( x_1 = 2 - x_2^2 \):

- Objective: \( f(x_2, x_3) = (2 - x_2^2) + 4x_2^2 + x_3^2.\)

- Constraint on \( x_1 \): \(-1 \leq 2 - x_2^2 \leq 1.\)
Example 7.1 (*continued*)

Minimize \( f(x) = (2 - x_2^2) + 4x_2^2 + x_3^2 \)

Subject to: \( 1 \leq x_2 \leq \sqrt{3} \)
\( 0 \leq x_3 \leq 2. \) \hspace{1cm} (2)

Not always possible to carry out substitution.
Equality constraints treatment.

Example 7.2

Minimize \[ f(x) = x_1 x_2 x_3 \]

Subject to:

\[ h_1(x) = x_1 + x_2 + x_3 - 1 = 0 \]
\[ h_2(x) = x_1^2 x_3 + x_2 x_3^2 + x_2^{-1} x_1 - 2 = 0 \]

\[ 0 \leq (x_1, x_3) \leq \frac{1}{2}. \]
Example 7.2 (continued)
Substitute \( x_1 = 1 - x_2 - x_3 \):

\[
\text{Minimize} \quad f(x) = (1 - x_2 - x_3)x_2x_3 \\
\text{Subject to:} \quad (1 - x_2 - x_3)^2x_3 + x_2x_3^2 + x_2^{-1}(1 - x_2 - x_3) - 2 = 0 \\
\quad 0 \leq (1 - x_2 - x_3) \leq \frac{1}{2} \\
\quad 0 \leq x_3 \leq \frac{1}{2}.
\]

Values of \( x_3 \) numerically calculated.
Outline

1. Introduction.
2. Problem preparation
3. Feasible starting points generation
4. Unconstrained search methods adaptation
5. Complex method
6. Random search
7. Conclusions
Starting points generation.

Only random generation taken into account.

- Pseudo-random generation routine:

\[ x_i = x_i^{(L)} + r_i(x_i^{(U)} - x_i^{(L)}), \text{ for } i = 1 \ldots N. \]  

(5)

With:
- \( r_i \in \mathbb{U}[0, 1] \)

Vector

\[ \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \]  

(6)

is then checked for feasibility.
Outline

1. Introduction.
2. Problem preparation
3. Feasible starting points generation
4. Unconstrained search methods adaptation
5. Complex method
6. Random search
7. Conclusions
Unconstrained search methods adaptation

Assumption:
- Equality constraints eliminated.

Introduction:
- Conjugate directions.
- Pattern search.

In-depth treatment of
- Complex method.
Conjugate directions

Assumptions:
- Given \( x^0 \) initial feasible point.
- Given set of search directions: \( d_i, i = 1 \ldots N \).

Figure 7.1. Premature termination of conjugate directions adaptation.

**Figure:** Premature termination of conj direction adapt.
Assumptions:

- Given $x^0$ initial feasible point.

Solution: retreat the step size.

Figure: Retraction of pattern step.
Pattern search: exploratory moves

Figure 7.3. Rejection of infeasible exploratory step.

Figure: Infeasible exploratory step rejection.

- If no better points found: step size reduced.
- Need for adaptive-directions that can be parallel to constraints.
Conjugate directions and pattern search

Consideration:

- Search directions need to be adjusted along constraints surfaces.

Possible solution:

- Randomized search directions.

Complex method

Simplex method summary:

- Generation and maintenance of search points.
- Undesirable points projection through centroids of remaining points.
Complex method keypoints:

- Set of $P$ points randomly generated.
- Each point tested for feasibility.
- If a point infeasible, retracted towards centroids of previous points.
- Objective function evaluated on the $P$ points.
- Worst point discarded.
- New point calculation through worst point projection.
- Search terminates if:
  - Points are sufficiently close.
  - Objective-function evaluations are close enough.
Complex method

Given a feasible point $x^0$, a reflection parameter $\alpha$, termination parameters $\epsilon$ and $delta$:

- Step 1: Generate initial set of $P$ feasible points. $\forall$ $p = 1, \ldots, P - 1$:
  - (a) Randomly generate $x^p_i$, $i = 1, \ldots, N$.
  - (b) if $x^p$ infeasible, calculate:
    
    $$x^p = x^p + \frac{1}{2}(\bar{x} - x^p).$$  
    
    repeat until $x^p$ becomes feasible.
  - (c) if $x^p$ is feasible, continue with (a) until $P$ points available.
  - (d) Evaluate, $\forall p = 1, \ldots, N$, $f(x^p)$. 

22 / 31
Complex method

- **Step 2: Reflection step.**
  - (a) Select the point $x^R$:
    \[
    f(x^R) = \max f(x^p)
    \]  
    (8)
  
  Set:
  \[
  F_{\text{max}} = f(x^R).
  \]  
  (9)

  (b) Calculate centroid $\bar{x}$ and:
    \[
    x^m = \bar{x} + \alpha(\bar{x} - x^R).
    \]  
  (10)

  (c) if $x^m$ feasible and $f(x^m) \geq F_{\text{max}}$, retract half the distance to the centroid $\bar{x}$. Continue until $f(x^m) < F_{\text{max}}$.  

  (d) if $x^m$ feasible and $f(x^m) < F_{\text{max}}$, go to step 4.  

  (e) if $x^m$ infeasible, go to step 3.
Complex method

- Step 3: Adjust for feasibility.
  
  (a) Bounds reset:
  - if \( x^m_i < x_i^{(L)} \), set \( x^m_i = x_i^{(L)} \).
  - if \( x^m_i > x_i^{(U)} \), set \( x^m_i = x_i^{(U)} \).

  (b) if \( x^m \) is infeasible, retract half the distance to the centroid. Continue until \( x^m \) is feasible, then go to step 2(c).
Step 4: Check for termination.

(a) Calculate:

\[ \bar{f} = \frac{1}{P} \sum f(x^p). \]  \hfill (11)

\[ \bar{x} = \frac{1}{P} \sum x^p. \]  \hfill (12)

(b) if:

\[ \frac{1}{P} \sum_p (f(x^p) - \bar{f})^2 \text{ and } \frac{1}{P} \sum_p \| (x^p - \bar{x}) \|^2 \leq \delta. \]  \hfill (13)

terminate. Otherwise go to step 2(a).
Example 7.3

Minimize \( f(x) = 30x_1x_2 + 30x_2x_3 + 60x_1x_3 \)

Subject to:

\( h_1(x) = x_1x_2x_3 - 16000 = 0 \)
\( g_1(x) = 110 - x_1 - x_2 \geq 0 \)
\( g_2(x) = 3x_1 - x_2 \geq 0 \)
\( g_3(x) = \frac{2}{3}x_2 - x_3 \geq 0 \)
\( 0 \leq x_1 \leq 60 \)
\( 0 \leq x_2 \leq 80 \)
\( 0 \leq x_3. \)
Example 7.3

Figure 7.4. Schematic for Example 7.3.

Figure: Schematic for example 7.3.
Example 7.3

Figure 7.5. Feasible region for Example 7.3.
Summary

Figure 7.6. Consequence of nonconvex feasible region: (a) centroid; (b) retraction.

Figure: Infeasible centroid and retraction for non-convex sets.

Complex method:
- Easy to implement.
- Requires convex feasible region.
- Distributed (multiuser) implementation.
Outline

1. Introduction.
2. Problem preparation
3. Feasible starting points generation
4. Unconstrained search methods adaptation
5. Complex method
6. Random search
7. Conclusions