# Engineering optimization.
# Chapter 7: Constrained direct search

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# Outline


1. Introduction.
2. Problem preparation
3. Feasible starting points generation
4. Unconstrained search methods adaptation
5. Complex method
6. Random search
7. Conclusions

# Introduction

- Scope: *constrained-optima localization*

Assumptions:

- Only function-values available.
- No gradient information.
- No analysis possible.

Difference from previous chapters:

- Explicit constraints consideration.

# Introduction

Two families of methods:

- Adaptation of unconstrained direct-search methods.
- Random selection of trail points.

Characteristics:

- Heuristic methods.
- Easy to implement.
- Slow convergence to optima.
- No guarantees on global optima.

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1 Introduction.

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# Problem preparation

Intuition:

- Given an initial feasible point $x^0$.
- Modify function-value based unconstrained localization methods to satisfy problem constraints.

Equality constraints issue:

- Assume points $x^0$ and $v$ satisfy constraints.
- Point: $x = x^0 + \alpha(v - x^0)$,with $0 \leq \alpha \leq 1$, need not satisfy constraints.

All direct-search methods adaptation require problems posed only in terms of inequality constraints.

# Equality constraints treatment

- Eliminate equality constraints by *substitution*.

Example 7.1

$$
\begin{aligned}
\underset{\mathbf{x}=[x_1,x_2,x_3]}{\text{Minimize}} \quad & f(\mathbf{x}) = x_1^2 + 4x_2^2 + x_3^2 \\
\text{Subject to:} \quad & h_1(\mathbf{x}) = x_1 + x_2^2 - 2 = 0 \\
& -1 \leq x_1 \leq 1 \\
& 0 \leq (x_2, x_3) \leq 2.
\end{aligned} \tag{1}
$$

Substitution of $x_1 = 2 - x_2^2$:

- Objective: $f(x_2, x_3) = (2 - x_2^2) + 4x_2^2 + x_3^2$.
- Constraint on $x_1$: $-1 \leq 2 - x_2^2 \leq 1$.

# Equality constraints treatment.

Example 7.1 (*continued*)

$$\begin{aligned} \underset{\mathbf{x}=[x_1,x_2,x_3]}{\text{Minimize}} \quad & f(\mathbf{x}) = (2 - x_2^2) + 4x_2^2 + x_3^2 \\ \text{Subject to:} \quad & 1 \le x_2 \le \sqrt{3} \\ & 0 \le x_3 \le 2. \end{aligned} \tag{2}$$

Not always possible to carry out substitution.

# Equality constraints treatment.

Example 7.2

$$
\begin{aligned}
\underset{\mathbf{x}=[x_1,x_2,x_3]}{\text{Minimize}} \quad & f(\mathbf{x}) = x_1 x_2 x_3 \\
\text{Subject to:} \quad & h_1(\mathbf{x}) = x_1 + x_2 + x_3 - 1 = 0 \\
& h_2(\mathbf{x}) = x_1^2 x_3 + x_2 x_3^2 + x_2^{-1} x_1 - 2 = 0 \\
& 0 \le (x_1, x_3) \le \frac{1}{2}.
\end{aligned} \tag{3}
$$

# Equality constraints treatment.

Example 7.2 (*continued*)

Substitute $x_1 = 1 - x_2 - x_3$:

$$\begin{aligned} \underset{\mathbf{x}=[x_1,x_2,x_3]}{\text{Minimize}} \quad & f(\mathbf{x}) = (1 - x_2 - x_3)x_2x_3 \\ \text{Subject to:} \quad & (1 - x_2 - x_3)^2 x_3 + x_2 x_3^2 + x_2^{-1}(1 - x_2 - x_3) - 2 = 0 \\ & 0 \leq (1 - x_2 - x_3) \leq \frac{1}{2} \\ & 0 \leq x_3 \leq \frac{1}{2}. \end{aligned} \tag{4}$$

Values of $x_3$ numerically calculated.

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# Starting points generation.

Only random generation taken into account.

- Pseudo-random generation routine:

$$x_i = x_i^{(\mathsf{L})} + r_i(x_i^{(\mathsf{U})} - x_i^{(\mathsf{L})}), \text{ for } i = 1 \ldots N. \tag{5}$$

With:

- $r_i \in \mathbb{U}[0, 1]$

Vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \tag{6}$$

is then checked for feasibility.

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# Unconstrained search methods adaptation

Assumption:

- Equality constraints eliminated.

Introduction:

- Conjugate directions.
- Pattern search.

In-depth treatment of

- Complex method.

# Conjugate directions

Assumptions:

- Given $x^0$ initial feasible point.
- Given set of search directions: $d_i$, $i = 1 \ldots N$.

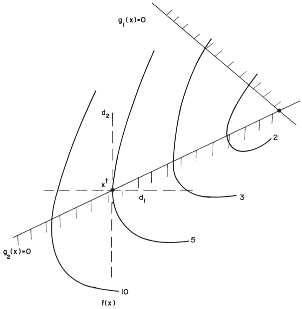


**Figure 7.1.** Premature termination of conjugate directions adaptation.

Figure: Premature termination of conj direction adapt.

# Pattern search

Assumptions:

- Given $x^0$ initial feasible point.

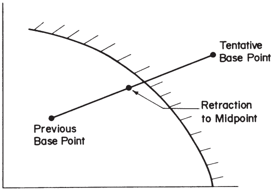


**Figure 7.2.** Retraction of pattern step.

Figure: Retraction of pattern step.

- Solution: retreat the step size.

# Pattern search: exploratory moves

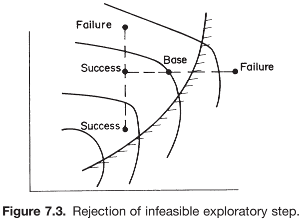


Figure: Infeasible exploratory step rejection.

- If no better points found: step size reduced.
- Need for adaptive-directions that can be parallel to constraints.

# Conjugate directions and pattern search

Consideration:

- Search directions need to be adjusted along constraints surfaces.

Possible solution:

- Randomized search directions.

Idea underlying simplex-method adaptation: *complex method*.

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# Complex method

Simplex method summary:

- Generation and maintenance of search points.
- Undesirable points projection through centroids of remaining points.

# Complex method

Complex method keypoints:

- Set of $P$ points randomly generated.
- Each point tested for feasibility.
- If a point infeasible, retracted towards centroids of previous points.
- Objective function evaluated on the $P$ points.
- Worst point discarded.
- New point calculation through worst point projection.
- Search terminates if:
  - Points are sufficiently close.
  - Objective-function evaluations are close enough.

# Complex method

Given a feasible point $x^0$, a reflection parameter $\alpha$, termination parameters $\epsilon$ and $delta$:

- Step 1: Generate initial set of $P$ feasible points. $\forall$ $p = 1, \ldots, P-1$:
  (a) Randomly generate $x_i^p$, $i = 1, \ldots, N$.
  (b) if $x^p$ infeasible, calculate:

$$x^p = x^p + \frac{1}{2}(\bar{x} - x^p). \tag{7}$$

  repeat until $x^p$ becomes feasible.
  (c) if $x^p$ is feasible, continue with (a) until $P$ points available.
  (d) Evaluate, $\forall$ $p = 1, \ldots, N$, $f(x^p)$.

# Complex method

- Step 2: Reflection step.
  (a) Select the point $x^R$:

$$f(x^R) = \max f(x^p) \tag{8}$$

  Set:

$$F_{\text{max}} = f(x^R). \tag{9}$$

  (b) Calculate centroid $\bar{x}$ and:

$$x^m = \bar{x} + \alpha(\bar{x} - x^R). \tag{10}$$

  (c) if $x^m$ feasible and $f(x^m) \geq F_{\text{max}}$, retract half the distance to the centroid $\bar{x}$. Continue until $f(x^m) < F_{\text{max}}$.
  (d) if $x^m$ feasible and $f(x^m) < F_{\text{max}}$, go to step 4.
  (e) if $x^m$ infeasible, go to step 3.

# Complex method

- Step 3: Adjust for feasibility.
  (a) Bounds reset:
    - if $x_i^m < x_i^{(L)}$, set $x_i^m = x_i^{(L)}$.
    - if $x_i^m > x_i^{(U)}$, set $x_i^m = x_i^{(U)}$.

(b) if $x^m$ is infeasible, retract half the distance to the centroid. Continue until $x^m$ is feasible, then go to step $2$(c).

# Complex method

- Step 4: Check for termination.
  (a) Calculate:

$$\bar{f} = \frac{1}{P} \sum f(x^p). \tag{11}$$

$$\bar{x} = \frac{1}{P} \sum x^p. \tag{12}$$

  (b) if:

$$\frac{1}{P} \sum_p (f(x^p) - \bar{f})^2 \text{ and } \frac{1}{P} \sum_p ||(x^p - \bar{x})||^2 \leq \delta. \tag{13}$$

  terminate. Otherwise go to step $2$(a).

# Example 7.3

$$
\begin{aligned}
\underset{\mathbf{x}=[x_1,x_2,x_3]}{\text{Minimize}} \quad & f(\mathbf{x}) = 30x_1x_2 + 30x_2x_3 + 60x_1x_3 \\
\text{Subject to:} \quad & h_1(\mathbf{x})x_1x_2x_3 - 16000 = 0 \\
& g_1(\mathbf{x}) = 110 - x_1 - x_2 \geq 0 \\
& g_2(\mathbf{x}) = 3x_1 - x_2 \geq 0 \\
& g_3(\mathbf{x}) = \frac{2}{3}x_2 - x_3 \geq 0 \\
& 0 \leq x_1 \leq 60 \\
& 0 \leq x_2 \leq 80 \\
& 0 \leq x_3.
\end{aligned} \tag{14}
$$

# Example 7.3

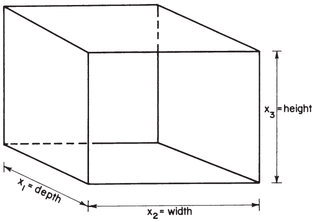


**Figure 7.4.** Schematic for Example 7.3.

Figure: Schematic for example 7.3.

# Example 7.3

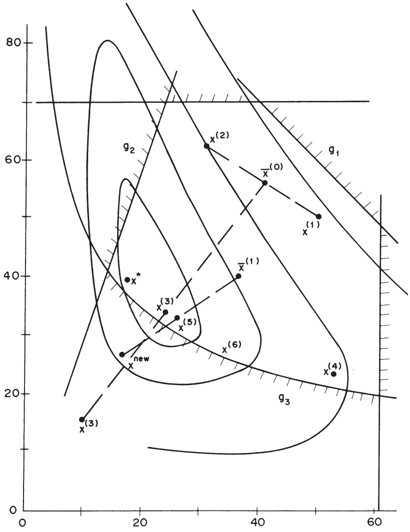


**Figure 7.5.** Feasible region for Example 7.3.

# Summary

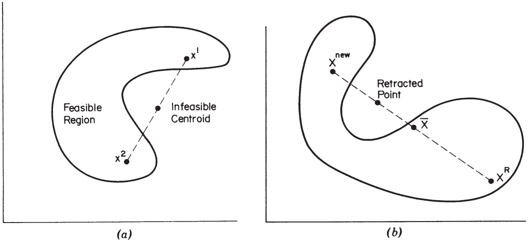


**Figure 7.6.** Consequence of nonconvex feasible region: (*a*) centroid; (*b*) retraction.

Figure: Infeasible centroid and retraction for non-convex sets.

Complex method:

- Easy to implement.
- Requires convex feasible region.
- Distributed (multiuser) implementation.

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