Topology, Capacity and Flow Assignment

Computer Communication Networks: Analysis and Design
(Klei. Vol. 2, Chap. 5)

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The problem

• The analysis of stochastic flows in packet networks is extremely complex

• We study some the more important problems arising in the design process

• Design variables:
  – Routing procedure (flow assignment)
  – Channel capacity
  – Topological configuration
  – Queuing discipline, packet numbering and sequencing, error control, etc..
Packet vs. Circuit Switching

Packet Switching

Time

Space

Propagation
Transmission
Processing
Queuing

End-to-end delay

\[ L/f_0 \]

\[ L_h/f_0 \]

\[ L_p/f_0 \]
A. Evolution of network structures

A.a. Private/most expensive (star network)
A.b. Least cost/slowest (minimal spanning tree)
Multidrop lines – polling or contention
A.c. Compromise/multiplexing or concentrating (statistical multiplexer)
A.d. Large network – combinations of (c)
B. Network modeling

M channels, N nodes, message (packet) switching

Assumptions:
(i) Links and nodes perfectly reliable (combinatorial in nature)
(ii) Nodal processing time negligible (for reading, error checking)
(iii) Nodal storage infinite
(iv) Fixed (or random) routing
(v) Offered traffic Poisson, packet length negative exponential
B.1. Arriving traffic

- Poisson with avg. rate [msg/sec]

\[ \gamma = \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_{jk} = \gamma_{jk} \]

\[ \gamma_{jk} \]

- total average flow entering in the system

\[ j \neq k \]

B.2. Messages

- Message lengths exponentially distributed with mean \[ \frac{1}{\mu} \] bits
B.3. Channel model

M/M/1 queue

Channel I

\[ C_i, \lambda_i \]

- \( C_i \) = channel capacity in bits/sec (bps)
- \( \lambda_i \) = avg. channel flow in msg/sec

- Note(1): \( \lambda = \sum_{i=1}^{M} \lambda_i \) Total average flow inside the network (\( \neq \gamma \) !)
- Note(2): propagation time \( P_i \)
B.4. Channel cost (in $)

\[ D = \sum_{i=1}^{M} d_i(C_i) \]

- \( d \) is a generic cost function
B.5. Average message delay

- $T = E [\text{message delay}]$
- $Z_{jk} \approx E [\text{message delay for message from } j \text{ to } k]$

$$T = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\gamma_{jk}}{\gamma} Z_{jk} = \frac{1}{\gamma} \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_{jk} Z_{jk}$$

fraction of traffic over $j$–$k$ node pair
C. Network design problems

C.1. Given: node locations, $\frac{1}{\mu}$, traffic matrix $\{\gamma_{jk}\}$

C.2. Objective function $T$

C.3. Parameters: $C_i$, $\lambda_i$, topology $\tau$

C.4. Constraint

$$\sum_{i=1}^{M} d_i(C_i) = D$$

4 optimization problems can be defined that differ only in the set of permissible design variables ->
C.3. The 4 problems are:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Given</th>
<th>Minimize w.r.t.</th>
<th>s.t.</th>
</tr>
</thead>
<tbody>
<tr>
<td>CA</td>
<td>$\lambda_i, \tau$</td>
<td>$T$</td>
<td>$C_i$</td>
</tr>
<tr>
<td>FA</td>
<td>$C_i, \tau$</td>
<td>$T$</td>
<td>$\lambda_i$</td>
</tr>
<tr>
<td>CFA</td>
<td>$\tau$</td>
<td>$T$</td>
<td>$C_i, \lambda_i$</td>
</tr>
<tr>
<td>TCFA</td>
<td>-</td>
<td>$T$</td>
<td>$C_i, \lambda_i, \tau$</td>
</tr>
</tbody>
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Increasing complexity

C.4. Dual problems

Swap $D$ with $T$

No dual for FA
D. Delay analysis

D.1. Model for $T$

Little’s result applied:

\[ \bar{N} = \gamma T = \sum_{i=1}^{M} \lambda_i T_i \]

\[ T = \frac{1}{\gamma} \sum_{i=1}^{M} \lambda_i T_i \]

$\lambda_i$ : one-way or two-way message flow

First take-away: global $T$ can be expressed as sum of local $T_i$
D.2. Independence assumption (a) and channel model (b)

D.2.a. Message lengths at nodes are independent at each node, and have exponentially distributed length $b$ with p.d.f.

\[ p(b) = \mu e^{-\mu b} \quad b \geq 0 \]

avg. message length = $\frac{1}{\mu}$ bits

- In fact, message lengths are not independent at each node since a message enters the network with a given length and retains that length from source to destination. This *independence assumption* is based on (i) large numbers of messages passing through each node and (ii) the moderate connectivity of the network, which support the assumption that packet lengths can be approximated as independent, exponentially-distributed message lengths.
D.2.b. M/M/1 channel model and T

Poisson arrivals: $\lambda_i$ messages/sec.

Exponential service: $\bar{x}_i = \frac{1}{\mu C_i}$ sec/message

$T_i = \frac{\bar{x}_i}{1 - \rho_i} = \frac{\bar{x}_i}{1 - \lambda_i \bar{x}_i} = \frac{1}{\mu C_i} = \frac{1}{\mu C_i - \lambda_i}$

$nb: [\mu C_i] = \frac{\text{message}}{\text{sec}}$
It follows that

$$T = \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\lambda_i}{\mu C_i - \lambda_i}$$

Other results (easily obtainable):

$$\rho_i = \frac{\lambda_i}{\mu C_i}$$

$$\overline{N}_i = \lambda_i T_i = \frac{\lambda_i}{\mu C_i - \lambda_i}$$

$$\overline{N}_{q_i} = \frac{\rho_i^2}{1 - \rho_i}$$

$$W_i = \frac{\lambda_i / \mu C_i}{\mu C_i - \lambda_i}$$
D.2.c. Other effects

Note: so far we neglected: 1. control traffic, 2. propagation and nodal processing delay

D.2.c.1. Average data message delay when control traffic is present

\[
\begin{align*}
\lambda_i &= \text{avg. flow of data messages} \\
\frac{1}{\lambda_i} &= \text{avg. length of data message} \\
\mu_i &= \text{avg. flow of all traffic} \\
\frac{1}{\mu_i} &= \text{avg. length of all messages} \\
\mu_i T_i &= \text{avg. data message delay} \\
\bar{x}_i &= \text{avg. service time for data messages} \\
W_i &= \text{avg. waiting time for all messages}
\end{align*}
\]

\(nb: \text{all traffic} = \text{data + control}\)
The avg. system time (with control traffic) can be now easily calculated as

\[ T_i = W_i + \bar{x}_i \]

\[ W_i = \frac{\lambda_i'}{\mu'C_i - \lambda_i'} \]

\[ \bar{x}_i = \frac{1}{\mu C_i} \]

\[ T_i = \frac{\lambda_i'}{\mu'C_i} + \frac{1}{\mu C_i} \]

\[ T = \sum_{i=1}^{M} \frac{\lambda_i}{\gamma} \left[ \frac{\lambda_i'}{\mu'C_i - \lambda_i'} + \frac{1}{\mu C_i} \right] \]
D.2.c.2. Propagation delay and nodal processing delay

\( P_i = \) propagation delay [sec/msg]

Depends upon the medium and the length of the link, e.g., ground station to geosynchronous satellite is on the order of 120 to 135 msec.

\( K = \) avg. nodal processing time [sec/msg]

\[
T = K + \sum_{i=1}^{M} \frac{\lambda_i}{\gamma} \left[ \frac{\lambda'_i}{\mu'C_i} + \frac{1}{\mu C_i} + P_i + K \right]
\]
A simple example
Question: How should $\gamma_{13}$ be split in order to minimize $T$?

Routing: all of $\gamma_{12}$ via $C_1$
1 of $\gamma_{13}$ via $C_1$ and $C_2$
5 of $\gamma_{13}$ via $C_3$
all of $\gamma_{23}$ via $C_2$

$\lambda_1 = 4 \quad \lambda_2 = 2 \quad \lambda_3 = 5 \quad \text{msg/sec}$

$\gamma = \gamma_{12} + \gamma_{13} + \gamma_{23} = 10 \quad \text{msg/sec}$

$T = \frac{1}{\gamma} \sum_{i=1}^{3} \frac{\lambda_i}{\mu C_i - \lambda_i} = \frac{1}{10} \left[ \frac{4}{12-4} + \frac{2}{6-2} + \frac{5}{9-5} \right]$

$T = 225 \quad \text{msec}$
D.2.d. Delay $T$ vs Offered load $\gamma$

Fixed routing

$$T = \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\lambda_i}{\mu C_i - \lambda_i}$$

D.2.d.1. [Low load] Decrease $\gamma$ such that $\lambda_i \ll \mu C_i \quad \forall \ i$

$$T \rightarrow \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\lambda_i}{\mu C_i}$$

So, no queuing in the network, all delay is due to service time (i.e., transmission delay)

$$T_0 \approx \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\lambda_i}{\mu C_i}$$
D.2.d.1. [High load] Increase $\gamma$ until some channel saturates, i.e., $\lambda_i \rightarrow \mu C_i$.

For that channel  $T_i \rightarrow \infty$, and, therefore so does $T \equiv T_i$.

Let $\gamma^*$ be the corresponding value of $\gamma$. 
D.2.e. Average path length

\[ \bar{\eta} = \text{avg. path length} \]

no. of links through which a message passes in proceeding from its source to its destination averaged over all source-destination node pairs
\[ \gamma = \gamma_{12} + \gamma_{13} + \gamma_{14} \]

\[ \overline{n} = \frac{\gamma_{12}.1 + \gamma_{13}.2 + \gamma_{14}.3}{\gamma} = \frac{\left( \gamma_{12} + \gamma_{13} + \gamma_{14} \right) + \left( \gamma_{13} + \gamma_{14} \right) + \gamma_{14}}{\gamma} \]

But \[ \lambda_1 = \gamma_{12} + \gamma_{13} + \gamma_{14} \]
\[ \lambda_2 = \gamma_{13} + \gamma_{14} \]
\[ \lambda_3 = \gamma_{14} \]

Hence, \[ \overline{n} = \frac{\lambda_1 + \lambda_2 + \lambda_3}{\gamma} = \frac{1}{\gamma} \sum_{i=1}^{3} \lambda_i \]

\[ \overline{n} = \frac{\lambda}{\gamma} \quad \text{where} \quad \lambda \cong \sum_{i=1}^{3} \lambda_i \]
In general,

\[ \pi_{jk} = \text{"path" from } j \text{ to } k \]

\[ n_{jk} = \text{no. of links in } \pi_{jk} \]

\[ \bar{n} = \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{j \neq k}^{N} \frac{\gamma_{jk} n_{jk}}{\gamma} \]
But, 
\[ \lambda \approx \sum_{i=1}^{M} \lambda_i = \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_{jk} n_{jk} \]

Since \( \gamma_{jk} \) msg/sec will traverse \( n_{jk} \) links in passing through the network

Hence,
\[ \bar{n} = \frac{\lambda}{\gamma} \quad \text{where} \quad \lambda = \sum_{i=1}^{M} \lambda_i \]

Using this result, we get
\[ T_0 = \bar{n} \sum_{i=1}^{M} \frac{\lambda_i}{\mu C_i} \]

for no-load delay with fixed routing.

See formula at pp. 22
Example: Same network as in delay analysis example above

\[ \gamma = 10 ; \quad \lambda_1 = 4 , \quad \lambda_2 = 2 , \quad \lambda_3 = 5 \]

So, \( \bar{n} = \frac{11}{10} = 1.1 \)

Also, using either expression for \( T_0 \), we have \( T_0 \approx 122.2 \) msec

This is the avg. msg. delay when the network is very lightly loaded.
E. Capacity Assignment (CA) Problem

Given: node locations, $\gamma_{jk}$, $\frac{1}{\mu}$, $\mathcal{T}$, $\{\lambda_i\}$;

$$\min \ T \ \text{w.r.t.} \ \{C_i\} \ \text{s.t.} \ \sum_{i=1}^{M} d_i(C_i) = D$$

\text{N.B.: the actual problem requires selection of } C_i \text{ from a finite set, e.g., } C_i \in \{100\text{Mbps, 1Gbps, etc.}\}, \text{ but, to find closed form solution, we have to relax this requirement}$$
E.1. Linear costs

Let us assume: \( d_i(C_i) = d_i C_i \quad i = 1, 2, \ldots, M \)

Then the optimization problem can be written as

\[
\min \quad T = \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\lambda_i}{\mu C_i - \lambda_i} \quad \text{s.t.} \quad \sum_{i=1}^{M} d_i C_i = D
\]

and it can solve using the Lagrangian:

\[
J = T + \beta \left[ \sum_{i=1}^{M} d_i C_i - D \right] = \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\lambda_i}{\mu C_i - \lambda_i} + \beta \left[ \sum_{i=1}^{M} d_i C_i - D \right]
\]

\[
\frac{\partial J}{\partial C_j} = 0 \quad j = 1, 2, \ldots, M
\]

\[
\frac{\partial J}{\partial C_j} = -\frac{\lambda_j \mu}{\gamma (\mu C_j - \lambda_j)^2} + \beta d_j = 0
\]
Solving the Lagrange’s equations:

\[
\left( \mu C_j - \lambda_j \right)^2 = \frac{\mu \lambda_i}{\beta \gamma d_j} \quad \rightarrow \quad C_j = \frac{\lambda_j}{\mu} + \frac{1}{\sqrt{\beta \gamma \mu}} \sqrt{\frac{\lambda_j}{d_j}}
\]

We can then find \( \beta \) by «forming the constraint»:

\[
D = \sum_{i=1}^{M} d_i C_i = \sum_{i=1}^{M} \frac{\lambda_i d_i}{\mu} + \frac{1}{\sqrt{\beta \gamma \mu}} \sum_{i=1}^{M} \sqrt{\lambda_i d_i}
\]

\[
\frac{1}{\sqrt{\beta \gamma \mu}} = \frac{D - \sum_{i=1}^{M} \frac{\lambda_i d_i}{\mu}}{\sum_{i=1}^{M} \sqrt{\lambda_i d_i}} \equiv \frac{D_e}{\sum_{i=1}^{M} \sqrt{\lambda_i d_i}}
\]

«Excess dollars»
Cont'd:

\[ C_j = \frac{\lambda_j}{\mu} + \frac{D_e}{d_j} \sum_{i=1}^{M} \sqrt{\lambda_i d_i} \]

\[ j = 1, 2, \ldots, M \quad (*) \]

\[ C_j = \{\text{min. required capacity to satisfy flow reqmnt.}\} + \]
\[ \{\text{sq. root allocation of normalized excess capacity}\} \]

If we substitute \( C_j \) in the o.f.

\[ T = \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\lambda_i}{\mu C_i - \lambda_i} \]

Note: from (*)

\[ \mu C_i - \lambda_i = \frac{\mu D_e}{d_i} \frac{\sqrt{\lambda_i d_i}}{\sum_{j=1}^{M} \sqrt{\lambda_j d_j}} \]
Cont’d:

\[ T_{\text{min}} = \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\lambda_i}{\mu D_e \sqrt{\lambda_i d_i}} = \frac{1}{\mu D_e \gamma} \left( \sum_{i=1}^{M} \sqrt{\lambda_i d_i} \right) \left( \sum_{j=1}^{M} \sqrt{\lambda_j d_j} \right) \]

\[ = \frac{1}{\mu D_e \gamma} \left[ \sum_{i=1}^{M} \sqrt{\lambda_i d_i} \right]^2 = \frac{\lambda}{\mu D_e \gamma} \left( \sum_{i=1}^{M} \sqrt{\frac{\lambda_i d_i}{\lambda}} \right)^2 \]

\[ T_{\text{min}} = \frac{n}{\mu D_e} \left( \sum_{i=1}^{M} \sqrt{\frac{\lambda_i d_i}{\lambda}} \right)^2 \]

Note: \( D_e \) must be greater than 0!
Cont’d:

Special case: \( d_i = 1 \) for all \( i \)

Hence, \( D = \sum_{i=1}^{M} C_i = C \) is the constraint

Using the definition of \( D_e \) at pp. 31

\[
D_e = C - \frac{1}{\mu} \sum_{i=1}^{M} \lambda_i = C - \frac{1}{\mu} \lambda
\]

\[
C_j = \frac{\lambda_j}{\mu} + D_e \frac{\sqrt{\lambda_j}}{\sum_{i=1}^{M} \sqrt{\lambda_i}}
\]

\[
T_{\text{min}} = \frac{1}{\mu D_e \gamma} \left( \sum_{i=1}^{M} \sqrt{\lambda_i} \right)^2
\]
Example:

**FLOWS**

\[ \lambda_1 = \lambda_2 = \lambda_4 = 4 \]
\[ \lambda_3 = 9 \quad \lambda_5 = \lambda_6 = 1 \]
\[ \rightarrow \lambda = 23 \]

**BUDGET** \((d_i=1)\) \(D=34\) (=C)

1 step: calculate **CAPACITIES**

\[ D_e = C - \frac{1}{\mu} \lambda = 11 \times 10^3 \left( \sum_{i=1}^{6} \sqrt{\lambda_i} \right) = 2 + 2 + 3 + 2 + 1 + 1 = 11 \]

\[ C_j = 1000\lambda_j + 11 \times 10^3 \frac{\sqrt{\lambda_j}}{11} \text{ bps} \quad \rightarrow C_j = \lambda_j + \sqrt{\lambda_j} \text{ (in kbps)} \]

\[ C_1 = C_2 = C_4 = 6\text{kbps} \quad C_5 = C_6 = 2\text{kbps} \quad C_3 = 12\text{kbps} \]

\[ C = \sum_i C_i = 34\text{kbps} \]

2 step: calculate **AVG DELAY**

\[ T_{\min} = \left( \frac{1}{\mu} \right) \frac{1}{D_e \gamma} \left( \sum_{i=1}^{6} \sqrt{\lambda_i} \right)^2 = 1000 \left( \frac{1}{11 \times 10^3} \right)^{16} (11)^2 \rightarrow \frac{11}{16} \text{ sec} = 687.5 \text{ m sec} \]
E.2. Other cost functions

Approximation of actual cost structures

a. Logarithmic cost (quantity discount)
\[ D = \sum_{i=1}^{M} d_i \log \alpha C_i \quad \text{CA is proportional} \]
\[ C_i = \frac{\lambda_i}{\mu} + b \lambda_i = a \lambda_i \]

b. Power law (ARPANET study - 1970)
\[ D = \sum_{i=1}^{M} d_i C_i^\alpha \quad 0 < \alpha < 1 \quad \alpha \approx 0.44 \]
\[ C_i - \frac{\mu}{\lambda_i} \left( \frac{1-\alpha}{\alpha} \right) C_i^2 = 0 \quad g_i = \left( \frac{\lambda_i}{\mu \gamma \alpha \beta d_i} \right)^{1/2} \]

\( \beta = \text{Lagrange multiplier (iterative solution required)} \)
E.3. Other forms of capacity assignment

- Heuristic approaches based on the notion of the network being “balanced” in some physically meaningful way

E.3.a. All links in the network have the same avg. utilization

\[ \rho_i = \frac{\lambda_i}{\mu C_i} = a \quad 0 < a < 1 \]

Fix “a” by specifying the desired avg. utilization, e.g., \( \rho_i = 0.6 \)

\[ C_i = \frac{1}{a} \frac{\lambda_i}{\mu} \]

Proportional CA

\[ T = \frac{Ma}{\gamma(1-a)} \]

\[ T = \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\lambda_i}{\mu C_i - \lambda_i} \]
E.3.b. All links in the network have the same link delay

\[ T_i = \frac{1}{\mu C_i - \lambda_i} = \tau \]

Simply, let us fix \( \tau \) by specifying the desired avg. delay for each link, (e.g., to \( T_i \) msec), then:

\[ C_i = \frac{\lambda_i}{\mu} + \frac{1}{\mu \tau} \]

\[ T = \bar{n} \tau \]
Example: \[ \frac{1}{\mu} = 200 \text{bits} \]

\[ \gamma_{12} = 3 \quad \gamma_{13} = 6 \quad \gamma_{23} = 1 \quad \text{[msg/sec]} \]

\[ \lambda_1 = 4 \quad \lambda_2 = 2 \quad \lambda_3 = 5 \quad \text{[msg/sec]} \]

Find capacities \( C_i \) so that all \( T_i \) are equal and \( T = 110 \text{ msec} \)

\[ \bar{n} = \frac{\lambda}{\gamma} = 1.1 \]

\[ \tau = \frac{T}{\bar{n}} = 100 \text{m sec} = 0.1 \text{sec} \]

\[ \frac{1}{\mu \tau} = 2000 \]

\[ C_1 = 2800 \text{bps} = 2.8 \text{kbps} \]

\[ C_2 = 2400 \text{bps} = 2.4 \text{kbps} \]

\[ C_3 = 3000 \text{bps} = 3.0 \text{kbps} \]
F. Flow Assignment (FA) problem

F.1. Given \( \tau \) and \( \{C_i\} \), minimize \( T \) with respect to \( \{\lambda_i\} \) subject to capacity constraints, \( 0 \leq \lambda_i \leq C_i \), external traffic requirements \( \{\gamma_{jk}\} \), and flow conservation

\[
T = \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\lambda_i}{C_i - \lambda_i}
\]
F.2. FA properties

1. Multicommodity-flow non-linear optimization problem
2. Closed-form solution not possible
3. Use iterative computational algorithm
4. Solution shall give both the amount of flows \( \{\lambda_i\} \) and the corresponding optimal (minimum delay) routing for all commodities
5. \( T=T(\lambda_1, \lambda_2 \ldots , \lambda_M) \) is a convex function of the \( \lambda_i \)'s
6. Capacity constraints, \( 0 \leq \lambda_i \leq C_i \), are convex
7. Fundamental theorem
   1. Any minimum of a convex function over a convex set is the global minimum
8. Best known and most effective algorithm for solving this problem is the \textit{Flow Deviation (FD) algorithm}
9. FD algorithm based on two basic notions:
   a. “Shortest path” flows
   b. Flow deviation which reduces minimization of \( T=T(\lambda_1, \lambda_2 \ldots , \lambda_M) \) over all the \( \lambda_i \) simultaneously to minimization of \( T(a) \) where \( a \) is a scalar, \( 0 \leq a \leq 1 \)
F.3. Shortest path flow part of FD

F.3.1. Link “length” \( l_i \) \( C_i \), \( \lambda_i \) \( l_i \)

\[
l_i = \frac{\partial T}{\partial \lambda_i} = \frac{C_i}{\gamma(C_i - \lambda_i)^2} > 0
\]

F.3.2. Problem statement

\[
\min_{\phi_i} \sum_{i=1}^{M} l_i \phi_i
\]

subject to capacity, total flow, and commodity flow constraints

(a) Linear programming problem
(b) Many efficient algorithms available
F.4. FD Algorithm

F.4.1. FD algorithm

n=iteration index

\( f(n) = (\lambda_1^n, \lambda_2^n, \ldots, \lambda_M^n) \) = flow vector on \( n^{th} \) iteration

\( f(0) = \) initial feasible flow (see Kleinrock vol. II, pp. 344-345)

a. Set \( n=0 \)

b. Solve shortest-path flow problem with weights \( l_i \)

and let \( \phi^{(n)} = [\phi_1^{(n)}, \phi_2^{(n)}, \ldots, \phi_M^{(n)}] \)

c. Set \( f^{(n+1)} = (1-a)f^{(n)} + a\phi^{(n)} \) and find \( 0 \leq a \leq 1 \) such that \( T(f^{(n+1)}) \)

is minimized

c. Any search method is ok, e.g., Fibonacci search

d. If \( T(f^{(n)}) - T(f^{(n+1)}) \leq \epsilon \) where \( \epsilon > 0 \) is the accuracy threshold, stop. Otherwise, set \( n = n+1 \) and return to Step 2.
F.4.2. FD algorithm flow chart

Test ($\varepsilon = \text{accuracy threshold}$)

(A) $T(f^{(n)}) - T(f^{n+1}) > \varepsilon$

(B) $T(f^{(n)}) - T(f^{n+1}) \leq \varepsilon$
F.5. FA Problem

F.5.1. Plausibility of FD algorithm

Assume $f^{(n)}$ given and let $f^{(n+1)} = (1-a)f^n + a\phi^n$

Then,

$$T(f^{(n+1)}) \approx T(f^{(n)}) + \sum_{i=1}^{M} \frac{\partial T}{\partial f_i}(f_i^{(n+1)} - f_i^{(n)})$$

and

$$T(f^{(n+1)}) \approx T(f^{(n)}) + a \sum_{i=1}^{M} l_i^{(n)}(\phi_i^{(n)} - f_i^{(n)})$$

Therefore, to minimize $T(f^{(n+1)})$ to within first order in $a$, it is necessary to choose the $\phi_i^{(n)}$ such that $\sum_{i=1}^{M} l_i^{(n)}\phi_i^{(n)}$ is minimized.
F.5.2. Problem set-up example

$$T^{(n)} = \frac{1}{\gamma} \sum_{i=1}^{4} \frac{\lambda_{i}^{(n)}}{C_{i} - \lambda_{i}^{(n)}}$$

$$\gamma = \gamma_{12} + \gamma_{13}$$

Conservation of total avg. flow:

$$\lambda_{1} + \lambda_{2} = \gamma_{12} + \gamma_{13}$$

$$\lambda_{1} - \lambda_{3} + \lambda_{4} = \gamma_{12}$$

$$\lambda_{2} + \lambda_{3} - \lambda_{4} = \gamma_{13}$$

If there were linearly independent, then the problem would be over.

$$\lambda_{1} + \lambda_{2} = \gamma_{12} + \gamma_{13}$$

$$\lambda_{1} - \lambda_{3} + \lambda_{4} = \gamma_{12}$$
a. Shortest path flow subproblem

\[
\min_{\phi_i} \sum_{i=1}^{4} l_i \phi_i \quad \text{where} \quad l_i = \frac{C_i}{\gamma(C_i - \lambda_i)^2}
\]

subject to ( \( \phi_i \) refers to \( \lambda_i \))

\[
\phi_1 + \phi_2 = \gamma_{12} + \gamma_{13} \quad \phi_1 - \phi_3 + \phi_4 = \gamma_{12}
\]

and

\[
0 \leq \phi_i \leq \beta C_i
\]

where \( \beta = 0.99 \) for example

b. FD algorithm gives the \( \lambda_i \) and \( T_{\text{min}} \), but not the routing
c. Extension to include commodity flow

Link 1: \( u_{12} \), \( u_{12} \)
\[ \lambda_1 = u_{12} + u_{12} \]

Link 2: \( u_{13} \), \( u_{13} \)
\[ \lambda_2 = u_{13} + u_{13} \]

Link 3: \( u_{23} \)
\[ \lambda_3 = u_{23} \]

Link 4: \( u_{32} \)
\[ \lambda_4 = u_{32} \]

Conservation of commodity flow:

\[ u_{12} + u_{13} = \gamma_{12} \quad u_{12} + u_{32} = \gamma_{12} \quad u_{13} + u_{23} = \gamma_{13} \]

\[ u_{13} + u_{13} = \gamma_{13} \quad u_{12} = u_{23} \quad u_{32} = u_{13} \]

6 eqns. of which 4 are linearly independent
c. Extension to include commodity flow (cont.)

Shortest path flow subproblem

\[ \phi_1 = u_{12}^{12} + u_{12}^{13} \quad \phi_2 = u_{13}^{12} + u_{13}^{13} \]

\[ \phi_3 = u_{23}^{13} \quad \phi_4 = u_{32}^{12} \]

\[ \min \sum_{i=1}^{4} l_i \phi_i \quad \text{where} \quad l_i = \frac{C_i}{\gamma(C_i - \lambda_i)^2} \]

Subject to

\[ u_{12}^{12} + u_{13}^{12} = \gamma_{12} \quad u_{12}^{13} + u_{13}^{13} = \gamma_{13} \quad u_{12}^{12} + u_{32}^{12} = \gamma_{12} \quad u_{13}^{13} + u_{23}^{13} = \gamma_{13} \]

and \[ 0 \leq \phi_i \leq \beta C_i \]

FD algorithm gives \( \lambda_i \), \( T_{\text{min}} \), \( u_{ab}^{jk} \)
G. CFA Problem

G.1. Given $\tau$, minimize $T$ with respect to $\{C_i\}$ and $\{\lambda_i\}$ subject to $\sum_{i=1}^{M} d_i C_i = D$ and $0 \leq \lambda_i \leq C_i$

where $T = \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\lambda_i}{C_i - \lambda_i}$

Note: When we combine CA and FA, we are no longer able to give globally optimal solutions, but rather describe procedures that find local minima for $T$
G.2. Approach

a. Combine CA and FA algorithms

b. Since \( \{C_i\} \)'s have to be assigned, it can be shown that fixed routing is optimal.
   - This implies setting \( a=1 \) in the FD algorithm

c. It can also be shown that shortest path routing gives only local minima for \( T \).
   - This means that several initial feasible flows \( f^{(0)} \) must be tried
G.3. Flow chart of CFA algorithm

- Feasible
- CA
- \( \{C_i^{(m)}\} \)
- FD (\( \alpha = 1 \))
- Test:
  - \( T(f^{(m+1)}) < T(f^{(m)}) \) (A)
  - \( T(f^{(m+1)}) \geq T(f^{(m)}) \) (B)
- Stop
G. 4. Algorithm properties

a. Converges for each \( f^{(0)} \) since the number of possible shortest-path flows is finite

b. Since \( l_i \geq 0 \), there are no negative loops in the shortest-path flow algorithm

\[
l_i = \frac{\partial T}{\partial (\lambda_i / \mu)} = \frac{n}{D_e} \left[ \sum_{j=1}^{M} \left( \frac{d_i}{\lambda \lambda_i} \right) + \frac{d_i}{\mu D_e} \sum_{j=1}^{M} \left( \frac{d_i}{\lambda \lambda_i} \right)^{1/2} \right] > 0
\]

c. The form of \( l_i \) that results from the “CA part” of algorithm is such that \( \lim_{\lambda_i \to 0} l_i = \infty \)

- See Kleinrock’s Vol. 2, Eq. 5.45
- It implies that once \( \lambda_i \) and \( C_i \) become zero, they remain there

d. Since only local minima are possible, one has to be clever in the \( f^{(0)} \) that are used
H. TCFA Problem (Dual Form)

H.1. Given node locations and \( \{\gamma_{jk}\} \), minimize

\[
D = \sum_{i=1}^{M} d_i(C_i)
\]

with respect to \( \tau \), \( \{C_i\} \) and \( \{\lambda_i\} \) subject to

\[
T = \sum_{i=1}^{M} \frac{\lambda_i}{C_i - \lambda_i} \leq T_{MAX}
\]

and connectivity constraints.

\[
0 \leq \lambda_i < C_i \quad i = 1,2,...M \quad 0 < M \leq N(N - 1)
\]

Note: \( M = \text{no. of links} = \text{variable!} \)

Note(2): in the slides \( \mu \) is not reported (..normalized)
H.2. Approach

a. Real design requires selection of discrete capacities

b. Heuristic solution procedure via iterative use of CFA and choices of $\tau$

c. Only local minima possible

d. $d_i(C_i) = \text{termination cost} + (\text{line cost}) \times (\text{length of line})$

expressed in $$/month

$$d_i(C_i) = a_i + d_iC_i^\alpha$$
H.3. TCFA$_2$ Problem

CBE (Concave Branch Elimination) Algorithm

a. Select $\tau^{(0)}$, e.g., fully-connected

b. For each channel, assume a power-law approximation and choose $d_i$ and $\alpha_i$ in

$$D = \sum_{i=0}^{M} d_i \alpha_i$$

c. Execute the CFA$_2$ (minimize $D$ subject to constraint $T=T_{\text{MAX}}$). At each iteration in CFA$_2$, use a linearized value for capacity about the current value of flow. If at any step in CFA$_2$, the connectivity constraint is violated, proceed to Step (d) with the previous step CFA$_2$ results; otherwise, let CFA$_2$ run to completion

d. Discretize the $\{C_i\}$ to the nearest actual line capacities available such that $\lambda_i < C_i$ and $T \leq T_{\text{MAX}}$
e. Refine the flow assignment using the dual form of the FD algorithm where

\[
l_i \equiv \frac{\partial D}{\partial \lambda_i} = d_i \left[ 1 + \frac{\sum_{j=1}^{M} \sqrt{\lambda_j d_j}}{\gamma T_{\text{MAX}} \sqrt{\lambda_i d_i}} \right]
\]

and adjust the \( \{C_i\} \) accordingly so that \( \lambda_i < C_i \) and \( T \leq T_{\text{MAX}} \).

f. Repeat Steps 3-5 for various feasible \( f^{(0)} \).

g. Repeat Steps 1-6 for a number of different \( \tau^{(0)} \).
a. $\gamma_{jk}$ 1kbps over all possible $jk$ pairs gives $\gamma=650$kbps
b. $T_{\text{MAX}}=200$ msec
c. Connectivity = 2
d. $D = \sum_{i=1}^{M} c_i + d_i C_i^\alpha$
e. Channel capacities and costs

<table>
<thead>
<tr>
<th>CAPACITY (KBPS)</th>
<th>TERMINATION COST ($/month)</th>
<th>LINE COST ($/month/mile)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.6</td>
<td>650</td>
<td>0.40</td>
</tr>
<tr>
<td>19.2 (2×9.6)*</td>
<td>1300</td>
<td>0.80</td>
</tr>
<tr>
<td>19.2</td>
<td>850</td>
<td>2.50</td>
</tr>
<tr>
<td>50</td>
<td>850</td>
<td>5.00</td>
</tr>
<tr>
<td>100 (2×50)*</td>
<td>1700</td>
<td>10.00</td>
</tr>
<tr>
<td>230.4</td>
<td>1350</td>
<td>30.00</td>
</tr>
</tbody>
</table>

Note: The total cost per month of a channel is given by:

\[
\text{total cost} = \text{termination cost} + (\text{line cost}) \times \text{length in miles}
\]

* Options obtained by using lower capacities in parallel.
f. Results

1) 30 initial topologies studied
2) CBE algorithm required 1 to 2 seconds CPU time on 360/91 per topology
3) $0.8 \leq \alpha_i \leq 1.0$ provided good fit to capacity costs and CBE worked well for these values
4) Two best solutions on next slides
Best Solution of ARPANET CBE
Best Solution of ARPANET CBE

a. Distribution of capacities vs. link lengths

<table>
<thead>
<tr>
<th>CAPACITY (KBPS)</th>
<th>LINK LENGTH (miles)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>&lt;100</td>
</tr>
<tr>
<td>9.6</td>
<td>0</td>
</tr>
<tr>
<td>19.2</td>
<td>1</td>
</tr>
<tr>
<td>50</td>
<td>11</td>
</tr>
</tbody>
</table>

Note: each entry represents the number of links having the specified capacity and lying within the specified length range.

b. $\tau^{(0)}$ was a fully-connected topology

c. Solution has 61 channels

d. Cost = $89,580/month
Second Best Solution of ARPANET CBE
Second Best Solution of ARPANET CBE (cont.)

a. $\tau^{(0)}$ was a low-connected topology

b. Both initial and final topologies had 29 channels

c. Uses higher speed lines (50 and 100kbps) for medium and long distances; 230kbps for short distances

d. Cost = $94,288/month
Convex Sets

- A set $S$ is said to be a *convex set* if, for any two points in the set, the line joining those two points is also in the set.

- Mathematically, $S$ is a convex set if for any two vectors $x^{(1)}$ and $x^{(2)}$ in $S$, the vector
  
  \[ x = \lambda x^{(1)} + (1 - \lambda)x^{(2)} \]

  is also in $S$ for any number between 0 and 1.

Examples:

- Figures A.1 and A.2 represent convex sets, while Figure A.3 is not a convex set.
The set of all feasible solutions to a linear programming problem is a convex set

The intersection of convex sets is a convex set (Figure A.4)

The union of convex sets is not necessarily a convex set (Figure A.4)

A hyperplane is a convex set

A half-space is a convex set

Figure A.4. Intersection and union of convex sets.
A convex combination of vectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$ is a vector $\mathbf{x}$ such that

$$x = \lambda_1 \mathbf{x}^{(1)} + \lambda_2 \mathbf{x}^{(2)} + \cdots + \lambda_k \mathbf{x}^{(k)}$$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$$

$$\lambda_i \geq 0 \text{ for } i = 1, 2, \ldots, k$$

An “extreme point” or “vertex” of a convex set is a point in the set that cannot be expressed as the midpoint of any two points in the set. For example, consider the convex set $\mathcal{S} = \{(x_1, x_2) \mid 0 \leq x_i \leq 2, \quad 0 \leq x_2 \leq 2\}$

This set has four extreme points given by $(0,0)$, $(0,2)$, $(2,0)$, and $(2,2)$.
A hyperplane is the set of all points $x$ satisfying $cx = z$ for a given vector $c \neq 0$ and a scalar $z$.

- The vector $c$ is called the normal to the hyperplane.
- For example, $H = \{(x_1, x_2, x_3) \mid 2x_1 - 3x_2 + x_3 = 5\}$ is a hyperplane.

A half-space is the set of all points $x$ satisfying $cx \leq z$ or $cx \geq z$ for a given vector $c \neq 0$ and a scalar $z$. 
Convex Function

A function of \( n \) variables \( f(x) \) defined on a convex set \( D \) is said to be a convex function if and only if for any two points \( x^{(1)} \) and \( x^{(2)} \in D \) and \( 0 \leq \lambda_i \leq 1 \),

\[
f[\lambda x^{(1)} + (1 - \lambda)x^{(2)}] \leq \lambda f(x^{(1)}) + (1 - \lambda)f(x^{(2)})
\]
Properties of Convex Functions

a. The chord joining any two points on the curve always falls entirely on or above the curve between those two points

b. The slope or first derivative of $f(x)$ is increasing or at least non-decreasing as $x$ increases

c. The second derivative of $f(x)$ is always non-negative for all $x$ in the interval

d. The linear approximation of $f(x)$ at any point in the interval always underestimates the true function value

e. For a convex function, a local minimum is always a global minimum
Figure A.6 illustrates property 4
Figure A.6 illustrates property 4

The linear approximation of $f(x)$ at the point $x^0$, denoted by $\tilde{f}(x; x^0)$, is obtained by ignoring the second and other higher order terms in the Taylor’s series expansion

$$\tilde{f}(x; x^0) = f(x^0) + \nabla f(x^0)(x - x^0)$$

For a convex function, property 4 implies that

$$f(x) \geq f(x^0) + \nabla f(x^0)(x - x^0) \text{ for all } x$$

The gradient of a function $f(x_1, \ldots, x_n)$ is given by

$$\nabla f(x_1, \ldots, x_n) = \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right]$$

The Hessian matrix of a function $f(x_1, \ldots, x_n)$ is an $n \times n$ symmetric matrix given by

$$H_f(x_1, \ldots, x_n) = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j} \right] = \nabla^2 f$$
Backup slides
Example:

\[ \gamma_{12} = 3 \quad \gamma_{13} = 6 \quad \gamma_{23} = 1 \ \text{[msg/sec]} \]

\[ \frac{1}{\mu} = 200 \text{bits} \]

\[ \lambda_1 = 4 \quad \lambda_2 = 2 \quad \lambda_3 = 5 \ \text{[msg/sec]} \]

Require all Ti to be the same and that

\[ \bar{n} = \frac{\lambda}{\gamma} = 1.1 \]

\[ \tau = \frac{T}{\bar{n}} = 100 \text{m sec} = 0.1 \text{sec} \]

\[ \frac{1}{\mu \tau} = 2000 \]

\[ T = 110 \text{msec} \]

\[ C_1 = 2800 \text{bps} = 2.8 \text{kbps} \]

\[ C_2 = 2400 \text{bps} = 2.4 \text{kbps} \]

\[ C_3 = 3000 \text{bps} = 3.0 \text{kbps} \]
FA Problem Example:

Ground network and satellite system

\[ C_{12} = 3 \quad C_{13} = 2.5 \quad C_{24} = 3 \quad C_{32} = 2 \quad C_{34} = 2 \quad C_{42} = 3 \]

\[ \lambda_{12} = 1.51 \quad \lambda_{13} = 0.49 \quad \lambda_{32} = 0.83 \]
\[ \lambda_{24} = 1.51 \quad \lambda_{34} = 0.66 \quad \lambda_{42} = 0.17 \]
\[ T = 1.51 \]

Determination of \( T \) includes 0.5 sec. Assumed propagation delay for each of the three satellite links.