Network Design and Planning (sq16)

Analysis of offered, carried and lost traffic in circuit-switched systems

Massimo Tornatore
Dept. of Electronics, Information and Bioengineering
Politecnico di Milano

Dept. Computer Science
University of California, Davis
tornator@elet.polimi.it
Summary

- General considerations
- Statistical traffic characterization
- Analysis of server groups
- Dimensioning server groups
Summary

- General considerations
  - Definitions
  - Parameters
- Traffic characterization
- Analysis of server groups
- Dimensioning server groups
Network

Basic concepts

- We are dealing with circuit-switched networks with given resources/capacity
- System that we analyse

![Diagram](image)

offered traffic/
users/sources

servers
Network

Basic concepts

- 3+(1) fundamental parameters
  - \( A \): offered load
  - \( m \): Service system with certain capacity
  - \( P \): quality of service (e.g. delay or blocking probability)
  - \( F \): functional characteristics (e.g. queueing discipline, routing technique, etc.)

- Problems
  - Dimensioning (synthesis, network planning)
    - Given \( A \), \( P \) (and \( F \)), find \( m \) at minimum cost/capacity
  - Performance evaluation (analysis)
    - Given \( A \), \( m \) (and \( F \)), find \( P \)
  - Management (traffic engineering)
    - Given \( A \) and \( m \), find \( F \) optimizing \( P \)
For each model a statistical characterization needed for
  - Traffic sources
  - Server systems

```
Traffic
sources
```

```
Service
system
```
Sources
- $S$ traffic sources
  - Generate connection requests (calls)
- Busy source: source engaged in a service request
  - Otherwise the user is not busy or free
- Average number of busy sources = Average amount of offered traffic

Servers
- $m$ system servers
  - Satisfy requests issued by sources
- Busy server: server engaged in a service to a source for a time duration requested by the source (holding time of the connection)
- Average number of busy servers = Average amount of carried traffic

Congestion: a connection request is not accepted $\Rightarrow$ Blocked request
- Denied request (loss systems)
- Delayed request (waiting systems)
Network

Basic concepts

- **E[θ]**: average holding time of a connection

- **Offered traffic**
  - \( \Lambda_o \): average rate of connection requests
  - \( A_o \): average number of connection requests issued in a time interval equal to the average holding time
    \[ A_o = \Lambda_o E[\theta] = \Lambda_o / \mu \]

- **Carried traffic**
  - \( \Lambda_s \): average acceptance rate of connection requests (statistical equilibrium)
  - \( A_s \): average number of connection requests accepted in a time interval equal to the average holding time
    \[ A_s = \Lambda_s E[\theta] = \Lambda_s / \mu \]

- **Lost traffic**
  - \( \Lambda_p \): average refusal rate of connection requests
  - \( A_p \): average number of connection requests denied in a time interval equal to the average holding time
    \[ A_p = \Lambda_p E[\theta] = \Lambda_p / \mu \]

- \( A_o, A_s, A_p \) adimensional \( \Rightarrow \) Erlang
How do we use queueing theory for traffic characterization?

- If $L=0$, and $m \to \infty$, then $A_o = A_s$
- This is the assumption we use for traffic characterization
Summary

- General considerations
- Traffic characterization
  - Statistical behaviour
  - Modeling of offered traffic
- Analysis of server groups
- Dimensioning server groups
Traffic theory

Traffic description

Statistical behaviour

- Relevant time instants
  - Time of service request
  - Time of service completion

- \( X(t, \omega) = \text{Number of servers busy at time } t \text{ of realization } \omega \text{ of the process} \)

- Assumptions
  - Stationarity
    - \( E_{[t_0,t_0+\tau]}[X(t, \omega)] = A_t(t_0, \omega) = A_t(\omega) = A(\omega) \)
  - Ergodicity
    - \( A(\omega) = A \)
Traffic theory

Traffic description

- Two main parameters
  - Holding time $\theta$ (duration of the call/request)
    - It is the inverse of the service rate: $E(\theta)=1/\mu$
    - We will stick to the traditional assumption of negative exponential distribution of the holding time
      - Simple and practical
  - Interarrival time $T$ (time between the arrival of two calls)
    - It is the inverse of the arrival rate $E(T)=1/\lambda$
    - We will consider the traditional assumption (Poisson), as well as two other cases (Bernoulli and Pascal)
- Possible histogram of holding times and corresponding approximation though exponential distribution

\[ \Pr\{\theta > t\} = e^{-\frac{t}{\theta}} \]
As for the interarrival time we will see three distributions:
- Pascal, Bernoulli, Poisson

Why are they interesting?
- See next slides
Traffic characterization

Poisson

- Parameters
  - $A_o = \Lambda_o = 30$
  - $m = 50$
Traffic characterization

Bernoulli

- Parameters
  - $A_o = 30$
  - $m = 50$
  - $S = 40$
Traffic characterization

Pascal

- Parameters
  - $A_o = 30$
  - $m = 50$
  - $c = 10$
Traffic theory

Traffic description

How do we model the three previous traffic behaviors?

- We use a birth & death process \([X(t)]\) to represent the offered traffic
  - Births: arrivals of service requests
  - Deaths: service completions
- In general b&d processes are characterized by two parameters
  - \(E[X(t)]\)
  - \(Var[X(t)]\) or \(VMR = \frac{Var[X(t)]}{E[X(t)]}\) (peakedness factor)
- Same characterization for all traffic types (offered, carried, lost)
- Typically, modelling simplicity suggests
  - Deaths: exponential - \(Pr[\theta > t] = e^{-t/\theta} = e^{-\mu t}\)
  - Births: Poisson - \(Pr[X(t) = k] = \frac{\lambda^k}{k!} e^{-\lambda t}\)
- In this lecture we go beyond Poisson (VMR = 1) and we also consider
  - Smoothed traffic (VMR < 1) - Bernoulli
  - Peaked traffic (VMR >1) - Pascal
Offered traffic model
Assumptions

- Arrival and service processes
  - Independent identically distributed (IID) interarrival times
  - IID service times
  - Arrival and service process mutually independent
  - Ergodicity
  - Stationarity
Offered traffic model

**Single source**

- **Source model**
  - Two states: idle (0) or busy (1)
    - \( \Pr\{0 \rightarrow 1 \text{ in } (t, t + \Delta t) | 0\} = \lambda' \Delta t \)
    - \( \Pr\{1 \rightarrow 0 \text{ in } (t, t + \Delta t) | 1\} = \mu \Delta t \)
  - \( \Rightarrow \) interarrival and service times with exponential distribution and
    - \( \lambda' \) = conditioned average interarrival rate (idle source)
    - \( \mu \) = conditioned average rate of service completion (busy source)
  - **Steady-state limiting probabilities**
    - \( q_0 = \frac{\mu}{\lambda' + \mu} \quad q_1 = \frac{\lambda'}{\lambda' + \mu} = A_o \)
    - \( 1 = \frac{1}{\lambda} = \frac{1}{\mu} + \frac{1}{\lambda'} \rightarrow \lambda = A_o \mu = \frac{\lambda' \mu}{\lambda' + \mu} \)
      - individual average interarrival rate
    - \( a = \frac{\lambda}{\mu} = q_1 = \frac{\lambda'}{\lambda' + \mu} = \frac{\alpha}{1 + \alpha} \)
      - offered traffic by a source
    - \( \alpha = \frac{\lambda'}{\mu} = \frac{q_1}{1 - q_1} = \frac{\lambda}{\mu - \lambda} = \frac{a}{1 - a} \)
      - offered traffic by an idle source
Offered traffic model

Multiple sources

- Single source model ensures that the occupancy process of a source groups is
  - markovian
  - continuous-time and time-homogeneous with discrete states
  - of birth & death type
- In formulas, this mean that the transition probabilities can be written as
  \[ \Pr\{0 \rightarrow 1 \text{ for a source in } (t, t + \Delta t) | n \text{ busy sources}, 0\} = \lambda'_n \Delta t \]
  \[ \Pr\{1 \rightarrow 0 \text{ for a source in } (t, t + \Delta t) | n \text{ busy sources}, 1\} = \mu_n \Delta t \]
- IID service times (also called source occupancy times) \( \Rightarrow \) \( \mu_n = n \mu \)
- Interbirth times described by three models
  - Bernoulli \( \lambda'_n = \lambda'(S - n) \) [S sources]
  - Poisson \( \lambda'_n = \Lambda_\infty = \lambda \) [\( \infty \) sources]
  - Pascal \( \lambda'_n = \lambda'(c + n) \) [\( \infty \) sources] [c integer]
Offered traffic model

Steady state characterization

- State probabilities in steady-state conditions derived by queues $M/M/\infty$
  - $X =$ Number of sources busy at the same time
  - $A_o = A_s = E[X] = \frac{\Lambda_o}{\mu}$

Bernoulli
$$p_n = \binom{S}{n} a^n (1-a)^{S-n}$$
$$n = 0, \ldots, S$$
$$a = p_1 = \frac{\lambda}{\mu} = \frac{\lambda'}{\lambda' + \mu}$$

Poisson
$$p_n = \frac{a^n}{n!} e^{-a}$$
$$a = \frac{\lambda}{\mu}$$

Pascal
$$p_n = \binom{c+n-1}{n} \alpha^n (1-\alpha)^c$$
$$\alpha = \frac{\lambda'}{\mu}$$
Offered traffic model

Steady state characterization

- State probabilities in steady-state conditions derived by queues M/M/∞
  - \( X = \text{Number of sources busy at the same time} \)
  - \( A_o = A_s = E[X] = \frac{\Lambda_o}{\mu} \)

<table>
<thead>
<tr>
<th>Bernoulli</th>
<th>Poisson</th>
<th>Pascal</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_n )</td>
<td>( \lambda' (S - n) )</td>
<td>( \lambda' (c + n) )</td>
</tr>
<tr>
<td>( \mu_n )</td>
<td>( n \mu )</td>
<td>( n \mu )</td>
</tr>
<tr>
<td>( \tilde{X} = A_o )</td>
<td>( Sa )</td>
<td>( \frac{c \alpha}{1 - \alpha} )</td>
</tr>
<tr>
<td>( \sigma_X^2 = \sigma_o^2 )</td>
<td>( Sa (1 - a) )</td>
<td>( \frac{c \alpha}{1 - \alpha} )</td>
</tr>
<tr>
<td>RVM</td>
<td>( 1 - a )</td>
<td>( \frac{1}{1 - \alpha} )</td>
</tr>
</tbody>
</table>

\[
\lambda_n = \Lambda_o + \lambda' (\tilde{X} - n) \\
\lambda_n = \Lambda_o + \lambda' (n - \tilde{X})
\]
Offered traffic model

Steady state characterization

- Traffic source models
  - Random traffic Poisson
  - Smoothed traffic Bernoulli
  - Peaked traffic Pascal

- Limiting cases
  - Bernoulli → Poisson with \( A_o = S \alpha \) if \( S \rightarrow \infty \) and \( \lambda' \rightarrow 0 \)
  - Pascal → Poisson with \( A_o = \frac{c \alpha}{1 - \alpha} \) if \( c \rightarrow \infty \) and \( \lambda' \rightarrow 0 \)

- Given \( E[X] \) and \( \text{Var}[X] \) one of the three traffic models is adopted with parameters

<table>
<thead>
<tr>
<th>Bernoulli</th>
<th>Poisson</th>
<th>Pascal</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S = \frac{A_o^2}{A_o - \sigma_X^2} = \frac{A_o}{1 - \text{RVM}} )</td>
<td>( A_o = \bar{X} = \sigma_X^2 )</td>
<td>( c = \frac{A_o^2}{\sigma_X^2 - A_o} = \frac{A_o}{\text{RVM} - 1} )</td>
</tr>
<tr>
<td>( a = 1 - \frac{\sigma_X^2}{A_o} = 1 - \text{RVM} )</td>
<td></td>
<td>( \alpha = 1 - \frac{A_o}{\sigma_X^2} = 1 - \frac{1}{\text{RVM}} )</td>
</tr>
</tbody>
</table>
Based on the previous 3 models for traffic sources, we can analyze the behaviour of the service system according to three main cases:

Behaviour of a source requesting service to a blocked system:

- Blocked calls **cleared** – BCC (chiamate perdute sparite - CPS) (loss system)
  - Source gives up

- Blocked calls **held** – BCH (chiamate perdute tenute - CPT)
  - Source keeps asking for service for a time $T_q$; $\theta_{\text{eff}} = \theta - T_q$

- Blocked calls **delayed** – BCD (chiamate perdute ritardate - CPR) (delay systems)
  - Sources keeps asking for service indefinitely
Summary

- General considerations
- Traffic characterization
- Analysis of server groups
  - Behavior upon congestion
  - Grade of service
- Dimensioning server groups
System with $m$ servers

- $X$: number of busy sources
- $n$: number of users in the system

CPS (BCC) ⇒ pure loss queue - M/M/m/0

- Time congestion: $S_p = \Pr\{\text{blocked system}\}$
- Call congestion: $\Pi_p = \Pr\{\text{blocked system} \mid 1 \text{ arrival}\}$

\[
S_p = \Pr\{X = m\} = \Pr\{n = m\}
\]

\[
\Pi_p = \Pr\{X = m \mid 1 \text{ arrival}\} = \frac{\Pr\{n = m \cap 1 \text{ arrival}\}}{\Pr\{1 \text{ arrival}\}}
\]

\[
= \frac{\Pr\{1 \text{ arrival} \mid n = m\} \Pr\{n = m\}}{\Pr\{1 \text{ arrival}\}} = S_p \frac{\Pr\{1 \text{ arrival} \mid n = m\}}{\Pr\{1 \text{ arrival}\}}
\]

- Poisson case: $\Pr\{1 \text{ arrival} \mid n = m\} = \Pr\{1 \text{ arrival}\} \Rightarrow S_p = \Pi_p$
Analysis of server group

Behaviour upon congestion - CPT

- System with \( m \) servers
  - \( X \): number of busy sources
  - \( n \): number of users in the system

- CPT (BCH) \( \Rightarrow \) queue with infinite servers of which \( m \) are true, the other fictitious – M/M/\( \infty \)
  - Source receives either true service (\( X < m \)) for the requested time or fictitious service (\( X = m \)) for a time \( T_q \)
  - Server becoming idle in state \( X = m \) makes effective a fictitious service for a residual time \( \theta_{\text{eff}} = \theta - T_q \)
  - Time congestion: \( S_t = \Pr\{\text{busy true servers}\} \)
  - Call congestion: \( \Pi_t = \Pr\{\text{busy true servers} | 1 \text{ arrival}\} \)
    \[
    S_t = \Pr\{X = m\} = \Pr\{n \geq m\} \\
    \Pi_t = \Pr\{X = m | 1\text{arr}\} = \Pr\{n \geq m | 1\text{arr}\} = \frac{\Pr\{n \geq m \cap 1\text{arr}\}}{\Pr\{1\text{arr}\}} \\
    \text{Poisson case: } \Pr\{1 \text{ arrival} | n = m\} = \Pr\{1 \text{ arrival}\} \Rightarrow S_t = \Pi_t
    \]
Analysis of server group

Behaviour upon congestion - CPR

- System with $m$ servers
  - $X$: number of busy sources
  - $n$: number of users in the system

- CPR (BCD) $\Rightarrow$ pure delay system - M/M/m
  - Time congestion: $S_r = \Pr\{\text{blocked service}\}$
  - Call congestion: $\Pi_r = \Pr\{\text{blocked service} \mid 1 \text{ arrival}\}$
    \[
    S_r = \Pr\{X = m\} = \Pr\{n \geq m\}
    \]
    \[
    \Pi_r = \Pr\{X = m \mid 1 \text{ arrival}\} = \frac{\Pr\{n \geq m \cap 1 \text{ arrival}\}}{\Pr\{1 \text{ arrival}\}}
    \]
  - Poisson case: $\Pr\{1 \text{ arrival} \mid n = m\} = \Pr\{1 \text{ arrival}\} \Rightarrow S_r = \Pi_r$
## Analysis of server group

**BCC (CPS) - \( m = \infty \)**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distr.</td>
<td>Bernoulli</td>
</tr>
<tr>
<td>Policy</td>
<td>CPS</td>
</tr>
<tr>
<td>Sim time</td>
<td>100</td>
</tr>
<tr>
<td>( m )</td>
<td>15</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>0.5</td>
</tr>
<tr>
<td>( \mu )</td>
<td>0.1</td>
</tr>
<tr>
<td>( s )</td>
<td>13</td>
</tr>
<tr>
<td>( A_0 )</td>
<td>11.1429</td>
</tr>
<tr>
<td>( \sigma^2 )</td>
<td>1.59184</td>
</tr>
<tr>
<td>RVM</td>
<td>0.142857</td>
</tr>
</tbody>
</table>
Analysis of server group

*BCC (CPS)*

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distr.</td>
<td>Bernoulli</td>
</tr>
<tr>
<td>Policy</td>
<td>CPS</td>
</tr>
<tr>
<td>Sim time</td>
<td>100</td>
</tr>
<tr>
<td>m</td>
<td>10</td>
</tr>
<tr>
<td>(\lambda)</td>
<td>0.5</td>
</tr>
<tr>
<td>(\mu)</td>
<td>0.1</td>
</tr>
<tr>
<td>S</td>
<td>13</td>
</tr>
<tr>
<td>(A_0)</td>
<td>11.1429</td>
</tr>
<tr>
<td>(\sigma^2)</td>
<td>1.59184</td>
</tr>
<tr>
<td>RVM</td>
<td>0.142857</td>
</tr>
</tbody>
</table>
Analysis of server group

BCH (CPT)

Parameter | Value
--- | ---
Distr. | Bernoulli
Policy | CPT
Sim time | 100
m | 10
\( \lambda \) | 0.5
\( \mu \) | 0.1
S | 13
\( A_0 \) | 11.1429
\( \sigma^2 \) | 1.59184
RVM | 0.142857

Traffic theory
Analysis of server group

BCD (CPR)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distr.</td>
<td>Bernoulli</td>
</tr>
<tr>
<td>Policy</td>
<td>CPR</td>
</tr>
<tr>
<td>Sim time</td>
<td>100</td>
</tr>
<tr>
<td>m</td>
<td>10</td>
</tr>
<tr>
<td>λ</td>
<td>0.5</td>
</tr>
<tr>
<td>μ</td>
<td>0.1</td>
</tr>
<tr>
<td>S</td>
<td>13</td>
</tr>
<tr>
<td>Ao</td>
<td>11.1429</td>
</tr>
<tr>
<td>σ²</td>
<td>1.59184</td>
</tr>
<tr>
<td>RVM</td>
<td>0.142857</td>
</tr>
</tbody>
</table>
In the next slides, the formulas for $S$ and $\Pi$ are reported for BCC, BCH and (partially) BCD.

Only a subset of the proofs is requested (according to what is covered during the lecture):

- No need to know the other formulas, we will only check the performance comparison on graphs.
- It is important to be able to interpret the two graphs.
- Note the observation about Molina’s formula.
Analysis of server group

**BCC (CPS) - Bernoulli**

\[
\lambda_n = \lambda'(S - n) \quad (n = 0, \ldots, m)
\]

\[
\mu_n = n\mu \quad (n = 1, \ldots, m)
\]

\[
p_n = \frac{\binom{S}{n}\alpha^n}{\sum_{j=0}^{m} \binom{S}{j}\alpha^j} \quad (n = 0, \ldots, m)
\]

\[
\alpha = \frac{\lambda'}{\mu} = \frac{a}{1-a(1-\Pi_p)} = \frac{\alpha}{1+\alpha\Pi_p} \quad (\text{free source}) ;
\]

\[
a = \frac{\lambda}{\mu} = \frac{\alpha}{1+\alpha(1-\Pi_p)} \quad (\text{source})
\]

\[
S_p = p_m
\]

\[
\Pi_p = S_p \frac{\lambda'(S - m)}{\Lambda_o} = \frac{\binom{S-1}{m}\alpha^m}{\sum_{j=0}^{m} \binom{S-1}{j}\alpha^j}
\]

\[
\Rightarrow \Pi_p(S, m, \hat{\alpha}) = S_p(S - 1, m, \hat{\alpha}) \quad (\text{Engset})
\]
Analysis of server group

**BCC (CPS) - Bernoulli**

\[
A_o = Sa \quad A_o = \frac{\Lambda_o}{\mu} = \frac{\sum_{j=0}^{m} \lambda_j p_j}{\mu} = \hat{\lambda}(S - \bar{n}) = \hat{\alpha}(S - A_s) = S\hat{\alpha} \quad \frac{\sum_{k=0}^{m} \binom{S-1}{k} \hat{\alpha}^k}{\sum_{j=0}^{m} \binom{S}{j} \hat{\alpha}^j}
\]

\[
A_s = A_o \left(1 - \Pi_p\right) = Sa \left(1 - \Pi_p\right) = \sum_{k=0}^{m} kp_k = S\hat{\alpha} \quad \frac{\sum_{k=0}^{m-1} \binom{S-1}{k} \hat{\alpha}^k}{\sum_{j=0}^{m} \binom{S}{j} \hat{\alpha}^j}
\]

\[
A_p = A_o - A_s = A_o \Pi_p
\]

\[
\sigma_s^2 = \sum_{k=0}^{m} (k - A_s)^2 p_k = A_s(S, m, \hat{\alpha}) \left[1 - \left[A_s(S, m, \hat{\alpha}) - A_s(S - 1, m - 1, \hat{\alpha})\right]\right]
\]

\[
\sigma_s^2 = Sa \left(1 - \Pi_p(S, m, \hat{\alpha})\right) \left[1 - \left[A_s(S, m, \hat{\alpha}) - A_s(S - 1, m - 1, \hat{\alpha})\right]\right]
\]

\[
\leq Sa \left[1 - \left[A_s(S, m, \hat{\alpha}) - A_s(S - 1, m - 1, \hat{\alpha})\right]\right] \leq Sa(1 - a) = \sigma_o^2
\]

\[
\Rightarrow \sigma_s^2 \leq \sigma_o^2
\]

\[
RVM = 1 - \left[A_s(S, m, \hat{\alpha}) - A_s(S - 1, m - 1, \hat{\alpha})\right] < 1 \quad \left(A_s(S, m, \hat{\alpha}) > A_s(S - 1, m - 1, \hat{\alpha})\right)
\]
Analysis of server group
BCC (CPS) - Poisson

\[ \lambda_n = \Lambda_0 = \lambda \quad (n = 0, \ldots, m) \]
\[ \mu_n = n\mu \quad (n = 1, \ldots, m) \]

\[ p_n = \frac{a^n}{n! \sum_{j=0}^{m} \frac{a^j}{j!}} \quad (n = 0, \ldots, m) \]

\[ S_\rho = \Pi_\rho = p_m = E_{1,m}(A_0) = \frac{a^m}{m! \sum_{j=0}^{m} \frac{a^j}{j!}} \quad \text{(Erlang B)} \]
Analysis of server group

BCC (CPS) - Poisson

\[ A_o = a = \frac{\lambda}{\mu} \]

\[ A_s = \sum_{k=0}^{m} kp_k = A_o \left( \frac{m}{k!} \sum_{j=0}^{m} \frac{A_o^k}{j!} \right) = A_o \left( 1 - \frac{A_o^m}{m!} \right) = A_o \left( 1 - E_{1,m}(A_o) \right) \]

\[ A_p = A_o E_{1,m}(A_o) \]

\[ \sigma_s^2 = \sum_{k=0}^{m} (k - A_s)^2 p_k = A_s(m) \left[ 1 - \left[ A_s(m) - A_s(m-1) \right] \right] \]

\[ < A_s(m) < A_o = \sigma_o^2 \left( A_s(m) > A_s(m-1) \right) \]

\[ \sigma_s^2 = A_s(m) \left[ 1 - A_o \left( E_{1,m-1}(A_o) - E_{1,m}(A_o) \right) \right] < A_o \left[ 1 - A_o \left( E_{1,m-1}(A_o) - E_{1,m}(A_o) \right) \right] < A_o = \sigma_o^2 \left( E_{1,m-1}(A_o) > E_{1,m}(A_o) \right) \]

\[ \Rightarrow \sigma_s^2 < \sigma_o^2 \]

\[ RVM = 1 - \left[ A_s(m) - A_s(m-1) \right] = 1 - A_o \left[ E_{1,m-1}(A_o) - E_{1,m}(A_o) \right] < 1 \]
Analysis of server group

Erlang-Engset comparison

- # users / # servers

**Engset**

**Erlang**

Obs. 1: the comparison is performed between Erlang and Engset fixing the same offered load $A_0$.

Obs. 2:
- For small #users, Erlang provides excessive sovraestimation.
- For large #users, the two formulas tend to return very similar results.
Analysis of server group
BCC (CPS) - Pascal

\[ \lambda_n = \hat{\lambda}'(c + n) \quad (n = 0, \ldots, m) \]
\[ \mu_n = n\mu \quad (n = 1, \ldots, m) \]

\[ \hat{\alpha} = \frac{\hat{\lambda}'}{\mu} = \frac{\alpha}{1 - \alpha \Pi_H} \]

\[ S_p = p_m = \frac{\left( \begin{array}{c} c + m \\ m \end{array} \right) \hat{\alpha}^m}{\sum_{j=0}^{m} \left( \begin{array}{c} c + j \\ j \end{array} \right) \hat{\alpha}^j} \]

\[ \Pi_p(c, m, \hat{\alpha}) = S_p(c + 1, m, \hat{\alpha}) \]

\[ p_n = \frac{\left( \begin{array}{c} c + n - 1 \\ n \end{array} \right) \hat{\alpha}^n}{\sum_{j=0}^{m} \left( \begin{array}{c} c + j - 1 \\ j \end{array} \right) \hat{\alpha}^j} \quad (n = 0, \ldots, m) \]
Analysis of server group

BCC (CPS) - Pascal

\[ A_o = \frac{c\alpha}{1-\alpha} \]
\[ \Lambda_o = \frac{\sum_{j=0}^{m} \lambda_j p_j}{\mu} = \frac{\hat{\lambda}'(c+\bar{X})}{\mu} = \frac{\hat{\lambda}'(c+A_s)}{\mu} = \hat{\alpha}(c+A_s) \]

\[ A_s = A_o (1 - \Pi_p) = \frac{c\alpha}{1-\alpha} (1 - \Pi_p) = \sum_{k=0}^{m} k p_k = c\hat{\alpha} \frac{\sum_{k=0}^{m-1} \binom{c+k}{k} \hat{\alpha}^k}{\sum_{j=0}^{m} \binom{c+j-1}{j} \hat{\alpha}^j} \]

\[ A_p = (A_o - A_s) = A_o \Pi_p \]

\[ \sigma_s^2 = \sum_{k=0}^{m} (k - A_s)^2 p_k = A_s(c,m,\hat{\alpha}) \left[ 1 + [A_s(c+1,m-1,\hat{\alpha}) - A_s(c,m,\hat{\alpha})] \right] \]

\[ \sigma_s^2 < \sigma_o^2 \] (definition)

\[ \text{RVM} = 1 + [A_s(c+1,m-1,\hat{\alpha}) - A_s(c,m,\hat{\alpha})] \leq 1 \]
Analysis of server group

\[ S_t = \sum_{k=m}^{S} p_k \]
\[ \Pi_t = \frac{1}{\Lambda_o} \sum_{k=m}^{S-1} \lambda_k p_k \]

- Bernoulli - Queue model M/M/S/0/S

\[ \lambda_n = \lambda'(S - n) \quad n = 0,\ldots,S \]
\[ \mu_n = n \mu \quad n = 1,\ldots, S \]
\[ p_n = \binom{S}{n} a^n (1-a)^{S-n} \quad n = 0,\ldots,S \]

\[ S_t = \sum_{k=m}^{S} \binom{S}{k} a^k (1-a)^{S-k} \]
\[ \Pi_t = \frac{1}{\sum_{k=m}^{S-1} \lambda(S - k)} \binom{S}{k} a^k (1-a)^{S-k} = \sum_{k=m}^{S-1} \binom{S-1}{k} a^k (1-a)^{S-k-1} \]
\[ \Pi_t(S, m, a) = S_t(S-1, m, a) \]
Analysis of server group

**BCH(CPT)**

- **Poisson** - Queue model $M/M/\infty$

\[
\lambda_n = \Lambda_0 = \lambda \\
\mu_n = n\mu \\
p_n = \frac{a^n}{n!} e^{-a} \\
S_t = \Pi_t = \sum_{k=m}^{\infty} \frac{a^k}{k!} e^{-a} \quad \text{(Molina)}
\]

- **Pascal** - Queue model $M/M/\infty$

\[
\lambda_n = \lambda'(c + n) \\
\mu_n = n\mu \\
p_n = \binom{c + n - 1}{n} \alpha^n (1 - \alpha)^c \\
S_t = \sum_{k=m}^{\infty} \binom{c + k - 1}{k} \alpha^k (1 - \alpha)^c \\
\Pi_t = \frac{\sum_{k=m}^{\infty} \lambda'(c + k) \binom{c + k - 1}{k} \alpha^k (1 - \alpha)^c}{c\alpha \frac{\mu}{1 - \alpha}} \\
\Pi_t(c, m, \alpha) = S_t(c + 1, m, \alpha)
\]
analysis of server group

BCD(CPR)

- **Bernoulli** - Queue model M/M/m/S - m/S

\[
\begin{align*}
\lambda_n &= \lambda'(S - n) \quad n = 0, \ldots, S \\
\mu_n &= \begin{cases} 
n\mu & n = 1, \ldots, m - 1 \\
m\mu & n = m, \ldots, S 
\end{cases} \\
p_n &= \begin{cases} 
\binom{S}{n} \alpha^n & n = 0, \ldots, m - 1 \\
\binom{S}{n} \alpha^n \frac{n!}{m! m^{n-m}} & n = m, \ldots, S 
\end{cases} \\
p_0^{-1} &= (1 + \alpha)^S + \sum_{k=m}^{S} \left( \binom{S}{k} \frac{k! m^{m-k}}{m!} - 1 \right) \alpha^k \\
S_r &= \sum_{k=m}^{S} p_k = \frac{p_m}{E_{1,S-m} \left( \frac{m}{\alpha} \right)} \\
\Pi_r &= \frac{1}{\Lambda_0} \sum_{k=m}^{S-1} \lambda_k p_k = \frac{S - m}{S - \tilde{n}} \frac{p_m}{E_{1,S-m-1} \left( \frac{m}{\alpha} \right)}
\end{align*}
\]
Analysis of server group

**BCD(CPR)**

- **Poisson - Queue model** $M/M/m$

\[
\lambda_n = \Lambda_0 = \lambda \quad n = 0, 1, \ldots
\]
\[
\mu_n = \begin{cases} 
  n\mu & n = 1, \ldots, m - 1 \\
  m\mu & n = m, m + 1, \ldots 
\end{cases}
\]
\[
p_n = \begin{cases} 
  p_0 a^n \frac{1}{n!} & n = 0, \ldots, m - 1 \\
  p_0 a^n \frac{1}{m! m^{n-m}} & n = m, m + 1, \ldots 
\end{cases}
\]

\[
p_0 = \left[ \sum_{k=0}^{m-1} \frac{a^k}{k!} + \frac{a^m}{m!} \frac{m}{m-a} \right]^{-1}
\]

\[
S_r = \Pi_r = \sum_{k=m}^{\infty} p_k = E_{2,m}(A_0) = \frac{a^m}{m!} \frac{m}{m-a} \frac{m}{m-a} \left( \sum_{k=0}^{m-1} \frac{a^k}{k!} + \frac{a^m}{m!} \frac{m}{m-a} \right) \quad (\text{Erlang C})
\]
Analysis of server group

BCC (CPS) – BCH (CPT) – BCD (CPR) comparison

Obs. 1: the comparison is performed between Erlang and Engset fixing the same offered load $A_o$. 

---

Traffic theory
Summary

- General considerations
- Traffic characterization
- Analysis of server groups
- Dimensioning server groups
  - Theoretical analysis
  - Wilkinson’s approach
  - Fredericks’ approach
  - Lindberger’s approach
Theoretical analysis
Overflow servers – Finite case

- Primary servers: \( n \)
- Overflow servers: \( m \)
- Sequential search

- Poisson offered traffic \( A_o = \sigma_o^2 = \frac{\lambda}{\mu} \)

- System described by a 2-d Markov chain with state \((j,i)\), \( j = 0,\ldots,n; \ i = 0,\ldots,m \)

\[
\begin{align*}
\lambda + (j + i)\mu p_{j,i} &= \lambda p_{j-1,i} + (j + 1)\mu p_{j+1,i} + (i + 1)\mu p_{j,i+1} \\
\lambda + (n + i)\mu p_{n,i} &= \lambda p_{n-1,i} + (i + 1)\mu p_{n,i+1} + \lambda p_{n,i-1} \\
(n + m)\mu p_{n,m} &= \lambda p_{n-1,m} + \lambda p_{n,m-1} \\
\sum_{j=0}^{n} \sum_{i=0}^{m} p_{j,i} &= 1
\end{align*}
\]
Theoretical analysis

Overflow servers – Finite case

\[ S_r^m(A_0) = \sum_{v=0}^{m} \frac{A_0^{m-v}}{(m-v)} \left( v + r - 1 \right) \left( \frac{v + r - 1}{v} \right) \]

\[ \frac{A_0^i}{j!} = \sum_{i=0}^{n} \frac{A_0^i}{j!} \quad E_{1,n}(A_0) = \frac{S_0^n}{S_1^n} \]

N.B.

\[ p_{j,i} = \sum_{x=0}^{m-i} (-1)^x K_{i+x} \binom{i+x}{i} S_{i+x}^{i-x} \quad (\text{Brockmeyer}) \]

\[ K_k = \sum_{r=k}^{m} (-1)^{r-k} \binom{r-1}{k-1} a_r \quad (k = 1, \ldots, m) \]

\[ K_0 = \frac{1}{S_1^{n+m}} \]

\[ a_r = \frac{1}{S_1^{n+m} S_r^n} \sum_{v=r}^{m} \binom{v-1}{r-1} S_0^{n+v} \quad (r = 1, \ldots, m) \]

- overflow servers

\[ Q_i = \sum_{j=0}^{n} p_{j,i} = \sum_{x=0}^{m-i} (-1)^x K_{i+x} \binom{i+x}{i} S_{i+1+x}^{n-x} \]

- primary servers

\[ P_j = \sum_{i=0}^{m} p_{j,i} = \frac{S_0^i}{S_1^n} \quad \text{Erlang-B per } j = n \]

Distributions
Theoretical analysis

Overflow servers – Finite case

Grade of service

\[ A_w = A_0 E_{1,n}(A_0) \quad A_p = A_0 E_{1,n+m}(A_0) \]

\[ A_s = \sum_{i=0}^{m} iQ_i = A_w - A_p = A_0 \left[ E_{1,n}(A_0) - E_{1,n+m}(A_0) \right] = A_0 \left( \frac{S_0^n}{S_1^n} - \frac{S_0^{n+m}}{S_1^{n+m}} \right) \]

\[ S_p(n) = \Pi_p(n) = E_{1,n}(A_0) \quad S_p(n+m) = \Pi_p(n+m) = E_{1,n+m}(A_0) \]

\[ S_p(m) = \frac{S_{m+1}^n}{S_m^n} E_{1,n+m}(A_0) \quad \Pi_p(m) = \frac{A_p}{A_w} = \frac{E_{1,n+m}(A_0)}{E_{1,n}(A_0)} \]
Theoretical analysis
Overflow servers – Infinite case

Overflow servers: \( m = \infty \)

- 2-d Markov chain (\( m = \infty \))

\[
\begin{align*}
[\lambda + (j + i)\mu] p_{j,i} &= \lambda p_{j-1,i} + (j + 1)\mu p_{j+1,i} + (i + 1)\mu p_{j,i+1} \\
[\lambda + (n + i)\mu] p_{n,i} &= \lambda p_{n-1,i} + (i + 1)\mu p_{n,i+1} + \lambda p_{n,i-1} \\
\sum_{j=0}^{n} \sum_{i=0}^{\infty} p_{j,i} &= 1
\end{align*}
\]

- Solution

\[
p_{j,i} = (-1)^j C_0^n \sum_{v=0}^{\infty} \binom{v}{i} \frac{(-A_o)^v}{v!} \frac{C_v^j}{C_{v+1}^n C_v^n}
\]

\[
C_v^h = e^{-A_o} \sum_{s=0}^{h} \binom{v+s-1}{s} \frac{A_o^{h-s}}{(h-s)} \quad h = 1,2,...
\]

\[
C_0^h = e^{-A_o} \frac{A_o^h}{h!}
\]
Theoretical analysis
Overflow servers – Infinite case

- \( m = \infty \rightarrow \text{carried traffic} = \text{offered traffic in overflow group} \)
- Factorial moments of \( Q_i \) give for overflow servers

\[
M_k = \sum_{i=0}^{\infty} \binom{i}{k} Q_i = A_o^k \frac{C_0^n}{C_k^n} \quad \text{(Riordan)}
\]

\[
\begin{align*}
[M_1 = A_o E_{1,n}(A_o)] & \Rightarrow A_w = M_1 = A_o E_{1,n}(A_o) \\
& \Rightarrow \sigma_w^2 = A_w \left[ 1 - A_w + \frac{A_o}{1 + n + A_w - A_o} \right]
\end{align*}
\]

(Wilkinson - 1956)
Analysis of channel group

**BCC (CPS)**

- **Channel group**
  - *m* channels
  - *A*<sub>o</sub> Poisson
  - BCC
    - \( A_s = A_o \left(1 - E_{1,m}(A_o)\right) \)
    - \( \sigma_s^2 = A_o A_s - m A_p + A_s - A_s^2 = A_s - A_o E_{1,m}(A_o) \left[m - A_s\right] \)
    - \( \text{RVM} = \frac{\sigma_s^2}{A_s} = 1 - A_o E_{1,m}(A_o) \frac{m - A_s}{A_s} < 1 \)
    - \( A_p = A_o E_{1,m}(A_o) \)
    - \( \sigma_p^2 = A_p \left[1 - A_p + \frac{A_o}{1 + m + A_p - A_o}\right] \)  \( \text{Wilkinson’s Formula} \)
    - \( \text{RVM} = \frac{\sigma_p^2}{A_p} = 1 - A_p + \frac{A_o}{1 + m + A_p - A_o} = 1 + \frac{-A_p - A_p m + A_p A_s + A_o}{1 + m - A_s} = \)
      \[= 1 + \frac{A_o - A_p - A_p (m - A_s)}{1 + (m - A_s)} = 1 + \frac{A_s - A_p (m - A_s)}{1 + (m - A_s)} = 1 + \frac{\sigma_s^2}{1 + (m - A_s)} > 1 \]
Dimensioning of overflow channels

Wilkinson approach

- Channel group loaded by $A_o$, $\sigma_o^2$ with RVM > 1

- Dimensioning/analysis
  1. Compute $A_{oe}$ and $m_e$ knowing $A_o$ e $\sigma_o^2$
     \[ A_o = A_{oe}E_{1,m_e}(A_{oe}) \]
     \[ \sigma_o^2 = A_o \left[ 1 - A_o + \frac{A_{oe}}{1 + m_e + A_o - A_{oe}} \right] \]
  2. Compute $m$ or $\Pi_p$ knowing $\Pi_p$ or $m$
     \[ A_p = A_o \Pi_p = A_{oe}E_{1,m_e+m}(A_{oe}) \]

$$A_{oe} = \sigma_{oe}^2$$
Dimensioning of overflow channels

**Erlang formula**

- **Erlang formula**

  \[ E_{1,m}(A_o) = \frac{A_o^m}{m!} \sum_{i=0}^{m} \frac{A_o^i}{i!} = E(m, A_o) \]

- **Recursive Erlang formula**

  \[ \left[ E_{1,m}(A_o) \right]^{-1} = \frac{m}{A_o} \left[ E_{1,m-1}(A_o) \right]^{-1} + 1 \quad m \text{ integer} \]

- **Erlang formula for } m \text{ real (Fortet representation)}**

  \[ \left[ E_{1,m}(A_o) \right]^{-1} = A_o \int_{0}^{\infty} e^{-A_o y} (1 + y)^m dy \quad m \text{ real} \]

- **Approximation of real } m \text{ value to the closest integer**
  - Ceiling: dimensioning of overflow group
  - Floor: dimensioning of equivalent group
Dimensioning of overflow channels

Overflow traffic - Average

Calculation based on Erlang-B formula

Average offered traffic, $A_o$ (Erlang)

Average lost traffic, $A_p$ (Erlang)

0 2 4 6 8 10 12 14

$10^{-2}$ $10^{-1}$ $10^0$ $10^1$

Formula di Erlang-B

Intensita media del traffico offerto, $A_o$ (Erlang)

Valore medio del traffico di trabocco, $A_p$ (Erlang)

Average offered traffic, $A_o$ (Erlang)
Dimensioning of overflow channels

Overflow traffic - Average

Calculation based on Erlang-B formula

Average offered traffic, $A_o$ (Erlang)

Average lost traffic, $A_p$ (Erlang)

Formula di Erlang-B

Intensità media del traffico offerto, $A_o$ (Erlang)

Valore medio del traffico di trabocco, $A_p$ (Erlang)

Calculation based on Erlang-B formula

$m = 0-50$
Dimensioning of overflow channels

Overflow traffic - Variance

Average offered traffic, $A_o$ (Erlang)

Variance of lost traffic, $A_p$ (Erlang)

---

M/M/m/0

Traffic theory
Dimensioning of overflow channels

Overflow traffic - Variance

Average offered traffic, $A_O$ (Erlang)

Variance of lost traffic, $A_p$ (Erlang)
Dimensioning of overflow channels

**Sum of multiple flows**

- Dimensioning of overflow traffic for \( n \) independent traffic flows
- \( Hp: \) statistical independence of flows \( A-B_i \) \( \Rightarrow \) independent overflow traffics
  \( \Rightarrow \) Offered traffic to \( A-C \) derived directly from sum of lost traffics
Dimensioning of overflow channels

Sum of multiple flows

Numerical solution for two Wilkinson equations \( \Rightarrow \) Rapp approximation

\[
A_{o_e} = \sigma_0^2 + 3z(z - 1) \quad z = \frac{\sigma_0^2}{A_o} \quad A_o = A_{o_e}E_{1, m_e}(A_{o_e})
\]

\[
A_p = A_o \Pi_p = A_{o_e}E_{1, m + m_e}(A_{o_e})
\]

\[
A_o = \sum_{i=1}^{n} A_{p_i} \quad \sigma_o^2 = \sum_{i=1}^{n} \sigma_{p_i}^2
\]
Dimensioning of channel groups

**Fredericks model**

- Applicable for a traffic $A_o$, $\sigma_o^2$ with arbitrary $z$ ($z = RVM <> 1$)
- Real situation
  - Group of $m$ channels
  - Births: individual with **general** interarrival distribution
- Model
  - $z$ integer, $z>1$
  - Births: in groups of $z$, with **exponential** interarrival distribution ($A_{og} = \sigma_{og}^2 = A_o$)
  - Deaths: in groups of $z$
  - $z$ channel groups with $m/z$ channels each

---

![Diagram showing general and exponential arrivals](image-url)

- General
- Exponential
- $\times$ 1 arrival
- $\square$ $z$ arrivals
Dimensioning of channel groups

Fredericks model

- $H_p$: $z$ channels requested per birth, resulting in one channel requested per group

$$A_{og} = \sigma_{og}^2 = A_o$$

$$A_{oz} = \sigma_{sz}^2 = A_{og}/z$$

$m/z$ $A_{sz}, \sigma_{sz}^2$ $\ldots$ $m/z$ $A_{sz}, \sigma_{sz}^2$ $\ldots$ $m/z$ $A_{sz}, \sigma_{sz}^2$
Fredericks model

Analysis

- Offered traffic in \( z \) flows has the first two moments of the real traffic
  - \( H_p: \) very small loss probability \( (A_s \cong A_o) \)
  - \( X = \) Total number of busy channels in the \( z \) groups
  - \( Y = \) Number of busy channels in each group

\[
\begin{align*}
E[Y] &= \text{Var}[Y] = \frac{A_{og}}{z} \\
X &= zY \\
E[X] &= zE[Y] = z \frac{A_{og}}{z} = A_o \\
\text{Var}[X] &= z^2 \text{Var}[Y] = z^2 \frac{A_{og}}{z} = zA_o = \sigma_o^2
\end{align*}
\]

- \( A_o = zA_{oz} \), \( A_s = zA_{sz} \), \( A_p = zA_{pz} \)
- Global loss probability = Individual group loss probability

\[
\Pi_p = E_{1/z} \left( \frac{A_o}{z} \right)
\]

- Fredericks model hold also for \( z \) real values and for \( z < 1 \)
Fredericks model

Analysis

\[ A_p = zA_{pz} = z\frac{A_o}{z} E_{1,\frac{m}{z}} \left( \frac{A_o}{z} \right) = A_o E_{1,\frac{m}{z}} \left( \frac{A_o}{z} \right) \]

\[ \sigma_p^2 = z^2 \sigma_{pz}^2 = z^2 A_{pz} \left( 1 - A_{pz} + \frac{A_{oz}}{1 + \frac{m}{z} + A_{pz} - A_{oz}} \right) \]

\[ = zA_p \left( 1 - \frac{A_p}{z} + \frac{A_o}{z + m + A_p - A_o} \right) \]

\[ A_s = zA_{sz} = z\frac{A_o}{z} \left[ 1 - E_{1,\frac{m}{z}} \left( \frac{A_o}{z} \right) \right] = A_o \left[ 1 - E_{1,\frac{m}{z}} \left( \frac{A_o}{z} \right) \right] \]

\[ \sigma_s^2 = z^2 \sigma_{sz}^2 = z^2 \left[ A_{sz} - A_{oz} E_{1,\frac{m}{z}} \left( \frac{A_o}{z} \right) \left( \frac{m}{z} - A_{sz} \right) \right] = z^2 \left[ \frac{A_s}{z} - A_{pz} \frac{m - A_s}{z} \right] \]

\[ = zA_s \left[ 1 - \frac{m - A_s}{zA_s} A_p \right] \]
Loss per flow
Lindberger model

- Offered traffic = sum of \( n \) flows
- Wilkinson-Fredericks models only give global average loss
- Lindberger model gives loss per flow
  - \( n \) flows statistical independent
  - \( z_i = \sigma_{oi}^2 / A_{oi} \) arbitrary (\( i = 1, \ldots, n \))
  - Arrivals of “heavy” calls, each requesting \( z_i \) channels, with exponential interarrivals
  - Equivalent to receiving a Poisson traffic \( A_{oi} / z_i \) (one request per time) on a group of \( m/z_i \) servers

\[ \Rightarrow \text{Same Fredericks equations if} \]
  - \( A_{oi}, \sigma_{oi}^2 \) non-Poisson traffic replaced by \( A_{ozi} = \sigma_{ozi}^2 = A_{oi} / z_i \)
  - Each request of the new flow \( i \) occupies \( z_i \) channels out of the \( m \) total channels
Lindberger model

Analysis

- $X_i$ : number of accepted requests for $i$-th flow
- $X$ : total number of busy channels $X = \sum_{i=1}^{n} X_i z_i$
- It is proven that
  \[
  \pi(k_1,\ldots,k_n) = \Pr\{X_1 = k_1,\ldots,X_n = k_n\} = \frac{1}{G} \prod_{i=1}^{n} \frac{A_{oz_i}^{k_i}}{k_i!} \left(\text{generalized Erlang formula}\right)
  \]
  \[
  \sum \pi(k_1,\ldots,k_n) = 1 \quad \text{(gives $G$)}
  \]
- Loss probability $\Pi_{pi} = \Pr\{X > m - z_i\}$
- Complex computation of distribution $\pi \Rightarrow$ approximation of $\pi$ such that
  \[
  \frac{\Pi_{pi}}{\Pi_{p}} = \frac{z_i}{z} = \frac{\frac{\sigma^2_{oi}}{A_{oi}}}{\frac{\sigma^2_{o}}{A_o}} = \frac{\frac{\sigma^2_{o}}{A_o}}{\sum_{i=1}^{n} \frac{\sigma^2_{oi}}{A_{oi}}} \quad \text{(stat. indip. of $n$ flows)}
  \]

Traffic theory
## Dimensioning/analysis of channel groups

### Overall equations

<table>
<thead>
<tr>
<th>Globale</th>
<th>Rivolo $i$ – esimo</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pi_p = E_{1,m} \left( \frac{A_o}{z} \right)$</td>
<td>$\Pi_{pi} = \Pi_p \frac{z_i}{z}$</td>
</tr>
<tr>
<td>$A_p = A_o \Pi_p$</td>
<td>$A_{pi} = A_{oi} \Pi_{pi}$</td>
</tr>
<tr>
<td>$\sigma_p^2 = zA_p \left( 1 - \frac{A_p}{z} + \frac{A_o}{z + m + A_p - A_o} \right)$</td>
<td>$\sigma_{pi}^2 = A_{pi} \left( 1 + \frac{z_p - 1}{\sigma_o^2} \sigma_{oi}^2 \right)$</td>
</tr>
<tr>
<td>$A_s = A_o (1 - \Pi_p)$</td>
<td>$A_{si} = A_{oi} (1 - \Pi_{pi})$</td>
</tr>
<tr>
<td>$\sigma_s^2 = zA_s \left[ 1 - m - A_s \frac{A_p}{zA_s} \right]$</td>
<td>$\sigma_{si}^2 = z_i A_{si} + \frac{A_s - m}{A_p} A_{pi}^2$</td>
</tr>
</tbody>
</table>

$$z = \frac{\sigma_o^2}{A_o} \quad z_i = \frac{\sigma_{oi}^2}{A_{oi}} \quad z_p = \frac{\sigma_p^2}{A_p}$$